

Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution

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1 Introduction

Let A be an associative unital algebra over a field k . The Hochschild cohomology $\mathrm{HH}^*(A)$ of A has a very rich structure. It is a graded commutative algebra via the cup product or the Yoneda product, and it has a graded Lie bracket of degree -1 so that it becomes a graded Lie algebra; these make $\mathrm{HH}^*(A)$ a Gerstenhaber algebra [1]. These structures have a good description in terms of bar resolution of A , but this resolution is huge and so is very useless for concrete computations.

The cup product is well studied. There are different formulas for computing it using an arbitrary projective resolution and they were used in many examples. The situation with the Lie bracket is more complicated. Almost all computation of it are based on the method of so-called comparison morphisms. This method allows to transfer elements of Hochschild cohomology from one resolution to another. For example, this method was applied for the description of the Lie bracket on the Hochschild cohomology of the group algebra of quaternion group of order 8 over a field of characteristic 2 in [2]. Later this method was applied for all local algebras of the generalized quaternion type over a field of characteristic 2.

Just a little time ago a formula for computing bracket via resolution, which is not a bar resolution, appeared in [4]. The proof given there is valid for a resolution that satisfies some conditions. Other formulas for the Lie bracket are proved in the current work. These formulas use chain maps from a resolution to its tensor powers and homotopies for some null homotopic maps defined by cocycles. Then the formula of [4] is slightly changed and proved for an arbitrary resolution. Further, we give some formulas for the Lie bracket using so-called contracting homotopies.

Then we discuss some formulas for the Connes' differential on the Hochschild homology. One of these formulas is a slight modification of the formula from [5]. Also we give a formula using contracting homotopies for the Connes' differential. Thus, in the case where the Connes' differential induces a BV structure on Hochschild cohomology, we obtain an alternative way for the computing of the Lie bracket. We discuss this in the case where the algebra under consideration is symmetric.

Finally, we give an example of an application of the discussed formulas. We describe the BV structure and the Gerstenhaber bracket on the Hochschild cohomology of one family of symmetric local algebras of the dihedral type. The Hochschild cohomology for these algebras was described in [6] and [7].

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2 Hochschild cohomology via the bar resolution

During this paper A always denotes some algebra over a field \mathbf{k} . We write simply \otimes instead of $\otimes_{\mathbf{k}}$.

Let us recall how to define the Hochschild cohomology, the cup product and the Lie bracket in terms of the bar resolution. The Hochschild cohomology groups are defined as $\mathrm{HH}^n(A) \cong \mathrm{Ext}_{A^e}^n(A, A)$ for $n \geq 0$, where $A^e = A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A .

Definition 1. An A^e -complex is a \mathbb{Z} -graded A -bimodule P with a differential of degree -1 , i.e. an A -bimodule P with some fixed A -bimodule direct sum decomposition $P = \bigoplus_{n \in \mathbb{Z}} P_n$ and an A -bimodule homomorphism $d_P : P \rightarrow P$ such that $d_P(P_n) \subset P_{n-1}$ and $d_P^2 = 0$. Let $d_{P,n}$ denote $d_P|_{P_n}$. The n -th homology of P is the vector space $H_n(P) = (\mathrm{Ker} d_{P,n})/(\mathrm{Im} d_{P,n+1})$. An A^e -complex P is called acyclic if $H_n(P) = 0$ for all $n \in \mathbb{Z}$. A map of A^e -complexes is a homomorphism of A -bimodules that respects the grading. If it also respects differential, it is called a *chain map*. A complex is called positive if $P_n = 0$ for $n < 0$. A pair (P, μ_P) is called a *resolution* of the algebra A if P is a positive complex, $H_n(P) = 0$ for $n > 0$ and $\mu_P : P_0 \rightarrow A$ is an A -bimodule homomorphism inducing an isomorphism $H_0(P) \cong A$.

Given an A -complex P , (P, A) denotes the \mathbf{k} -complex $\bigoplus_{n \leq 0} \mathrm{Hom}_{A^e}(P_{-n}, A)$ with differential $d_{(P,A),n} = \mathrm{Hom}_{A^e}(d_{P,-1-n}, A)$. Let $\mu_A : A \otimes A \rightarrow A$ be the multiplication map.

Let $\mathrm{Bar}(A)$ be the positive A^e -complex with n -th member $\mathrm{Bar}_n(A) = A^{\otimes(n+2)}$ for $n \geq 0$ and the differential $d_{\mathrm{Bar}(A)}$ defined by the equality

$$d_{\mathrm{Bar}(A)}(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1}$$

for $n > 0$ and $a_i \in A$ ($0 \leq i \leq n+1$). Then $(\mathrm{Bar}(A), \mu_A)$, is a projective A^e -resolution of A that is called the bar resolution.

The Hochschild cohomology of the algebra A is the homology of the complex $C(A) = (\mathrm{Bar}(A), A)$. We write $C^n(A)$ instead of $C_{-n}(A)$ and δ^n instead of $d_{C(A),-1-n}$. Note that $C^0(A) \simeq A$ and $C^n(A) \simeq \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n}, A)$. Given $f \in C^n(A)$, we introduce the notation

$$\delta_n^i(f)(a_1 \otimes \cdots \otimes a_{n+1}) := \begin{cases} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}), & \text{if } i = 0, \\ (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}), & \text{if } 1 \leq i \leq n, \\ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}, & \text{if } i = n+1. \end{cases}$$

Then $\delta^n = \sum_{i=0}^{n+1} \delta_n^i$. We have $\mathrm{HH}^n(A) = (\mathrm{Ker} \delta^n)/(\mathrm{Im} \delta^{n-1})$.

The cup product $\alpha \smile \beta \in C^{n+m}(A) = \mathrm{Hom}_{\mathbf{k}}(A^{\otimes(n+m)}, A)$ of $\alpha \in C^n(A)$ and $\beta \in C^m(A)$ is given by

$$(\alpha \smile \beta)(a_1 \otimes \cdots \otimes a_{n+m}) := \alpha(a_1 \otimes \cdots \otimes a_n) \beta(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

This cup product induces a well-defined product in the Hochschild cohomology

$$\smile: \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m}(A)$$

that turns the graded \mathbf{k} -vector space $\mathrm{HH}^*(A) = \bigoplus_{n \geq 0} \mathrm{HH}^n(A)$ into a graded commutative algebra ([1, Corollary 1]).

The Lie bracket is defined as follows. Let $\alpha \in C^n(A)$ and $\beta \in C^m(A)$. If $n, m \geq 1$, then, for $1 \leq i \leq n$, we define $\alpha \circ_i \beta \in C^{n+m-1}(A)$ by the equality

$$(\alpha \circ_i \beta)(a_1 \otimes \cdots \otimes a_{n+m-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1});$$

if $n \geq 1$ and $m = 0$, then $\beta \in A$ and, for $1 \leq i \leq n$, we set

$$(\alpha \circ_i \beta)(a_1 \otimes \cdots \otimes a_{n-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta \otimes a_i \otimes \cdots \otimes a_{n-1});$$

for any other case, we set $\alpha \circ_i \beta$ to be zero. Now we define

$$\alpha \circ \beta := \sum_{i=1}^n (-1)^{(m-1)(i-1)} \alpha \circ_i \beta \text{ and } [\alpha, \beta] := \alpha \circ \beta - (-1)^{(n-1)(m-1)} \beta \circ \alpha.$$

Note that $[\alpha, \beta] \in C^{n+m-1}(A)$. The operation $[\ , \]$ induces a well-defined Lie bracket on the Hochschild cohomology

$$[\ , \]: \mathrm{HH}^n(A) \times \mathrm{HH}^m(A) \longrightarrow \mathrm{HH}^{n+m-1}(A)$$

such that $(\mathrm{HH}^*(A), \smile, [\ , \])$ is a Gerstenhaber algebra (see [1]).

3 Comparison morphisms

Here we recall the method of comparison morphisms. But firstly we introduce some notation.

If P is a complex, then we denote by $P[t]$ the complex, which equals to P as an A -bimodule, with grading $P[t]_n = P_{t+n}$ and differential defined as $d_{P[t]} = (-1)^t d_P$. Note that d_P defines a map from P to $P[-1]$. Let now take some map of complexes $f: P \rightarrow Q$. If $t \in \mathbb{Z}$, then $f[t]$ denotes the map from $P[t]$ to $Q[t]$ induced by f , i.e. such a map that $f[t]|_{P[t]_i} = f|_{P_{i+t}}$. For simplicity we will write simply f instead of $f[t]$, since in each situation t can be easily recovered. Let $\mathbf{d}f$ denote the map $f d_P - d_Q f: P \rightarrow Q[-1]$. For two maps of complexes $f, g: P \rightarrow Q$ we write $f \sim g$ if $f - g = \mathbf{d}s$ for some $s: U \rightarrow V[1]$. Note that if $f \sim 0$ and $\mathbf{d}g = 0$, then $fg \sim 0$ and $gf \sim 0$ (for the composition that has sense). Also we always identify an A -bimodule M with the complex \tilde{M} such that $\tilde{M}_i = 0$ for $i \neq 0$ and $\tilde{M}_0 = M$. Note also that if $f \sim 0$, then $\mathbf{d}f = 0$. It is not hard to see that if P is a projective complex, Q is exact in Q_i for $i \geq n$, and $Q_i = 0$ for $i < n$, then for any $f: P \rightarrow Q$ the equality $\mathbf{d}f = 0$ holds iff $f \sim 0$. Moreover, we have the following fact.

Lemma 1. *Let P be a projective complex, Q be exact in Q_i for $i > n$, and $Q_i = 0$ for $i < n$. Let $\mu_Q: Q \rightarrow H_n(Q)$ denote the canonical projection. If $f: P \rightarrow Q$ is such that $\mathbf{d}f = 0$ and $\mu_Q f \sim 0$, then $f \sim 0$.*

Proof. Assume that $\mu_Q f = \phi d_P$. Since P_{n-1} is projective, there is some $\psi : P_{n-1} \rightarrow Q_n$ such that $\mu_Q \psi = \phi$. Then $f - \mathbf{d}\psi$ is a chain map such that $\mu_Q(f - \mathbf{d}\psi) = 0$. Then it is easy to see that $f \sim \mathbf{d}\psi \sim 0$.

□

Let now (P, μ_P) and (Q, μ_Q) be two A^e -projective resolutions of A . The method of comparison morphisms is based on the following idea. Since P is positive projective and Q is exact in Q_i for $i > 0$, there is some chain map of complexes $\Phi_P^Q : P \rightarrow Q$ such that $\mu_Q \Phi_P^Q = \mu_P$. Analogously there is a chain map $\Phi_Q^P : Q \rightarrow P$ such that $\mu_P \Phi_Q^P = \mu_Q$. Then Φ_P^Q and Φ_Q^P induce maps from (Q, A) to (P, A) and backwards. Thus, we also have the maps

$$(\Phi_P^Q)^* : H_*(Q, A) \rightarrow H_*(P, A) \text{ and } (\Phi_Q^P)^* : H_*(P, A) \rightarrow H_*(Q, A).$$

Since $\mathbf{d}(1_P - \Phi_Q^P \Phi_P^Q) = 0$, we have $1_P \sim \Phi_Q^P \Phi_P^Q$ by the arguments above. Then it is easy to see that $(\Phi_P^Q)^*(\Phi_Q^P)^* = (\Phi_Q^P \Phi_P^Q)^* = 1_{H_*(P, A)}$ and, analogously, $(\Phi_Q^P)^*(\Phi_P^Q)^* = 1_{H_*(Q, A)}$. So we can define the Hochschild cohomology of A as the homology of (P, A) , and this definition does not depend on the A^e -projective resolution (P, μ_P) of A . If we define some bilinear operation $*$ on (Q, A) , which induces an operation on $\mathrm{HH}^*(A)$, then we can define the operation $*_\Phi$ on (P, A) by the formula $f *_\Phi g = (f \Phi_Q^P * g \Phi_Q^P) \Phi_P^Q$ for $f, g \in (P, A)$. It is easy to see that $*_\Phi$ induces an operation on $\mathrm{HH}^*(A)$ and that the induced operation coincides with $*$. Now we can take $Q = \mathrm{Bar}(A)$ and define the cup product and the Lie bracket on (P, A) by the equalities

$$f \smile_\Phi g = (f \Phi_{\mathrm{Bar}(A)}^P \smile g \Phi_{\mathrm{Bar}(A)}^P) \Phi_P^{\mathrm{Bar}(A)} \text{ and } [f, g]_\Phi = [f \Phi_{\mathrm{Bar}(A)}^P, g \Phi_{\mathrm{Bar}(A)}^P] \Phi_P^{\mathrm{Bar}(A)}.$$

Thus, to apply the method of comparison morphism one has to describe the maps $\Phi_P^{\mathrm{Bar}(A)}$ and $\Phi_{\mathrm{Bar}(A)}^P$ and then use them to describe the bracket in terms of the resolution P . The problem is that for some $x \in P$ the formula $\Phi_P^{\mathrm{Bar}(A)}(x)$ is complicated and that to describe $\Phi_{\mathrm{Bar}(A)}^P$ one has to define it on a lot of elements.

Let now recall one formula for the cup product that uses an arbitrary A^e -projective resolution of A instead of the bar resolution. But firstly let us introduce some definitions and notation.

Definition 2. Given A^e -complexes P and Q , we define the *tensor product complex* $P \otimes_A Q$ by the equality $(P \otimes_A Q)_n = \sum_{i+j=n} P_i \otimes_A Q_j$. The differential $d_{P \otimes Q}$ is defined by the equality $d_{P \otimes Q}(x \otimes y) = d_P(x) \otimes y + (-1)^i x \otimes d_Q(y)$ for $x \in P_i$, $y \in Q_j$.

We always identify $P \otimes A$ and $A \otimes P$ with P by the obvious isomorphisms of complexes. For any $n \in \mathbb{Z}$ we also identify $P \otimes_A Q[n]$ and $P[n] \otimes_A Q$ with $(P \otimes_A Q)[n]$. Note that this identification uses isomorphisms $\alpha_{P, Q}^n : P \otimes_A Q[n] \rightarrow (P \otimes Q)[n]$ and $\beta_{P, Q}^n : P[n] \otimes_A Q \rightarrow (P \otimes Q)[n]$ defined by the equalities $\alpha_{P, Q}^n(x \otimes y) = (-1)^{in} x \otimes y$ and $\beta_{P, Q}^n(x \otimes y) = x \otimes y$ for $x \in P_i$ and $y \in Q$. In particular, we have two different isomorphisms $\beta_{P, Q}^n \alpha_{P[n], Q}^m$ and $\alpha_{P, Q}^m \beta_{P, Q[m]}^n$ from $P[n] \otimes_A Q[m]$ to $(P \otimes Q)[n + m]$. For convenience, we always identify $P[n] \otimes_A Q[m]$ with $(P \otimes Q)[n + m]$ using the isomorphism $\beta_{P, Q}^n \alpha_{P[n], Q}^m$ that sends $x \otimes y$ to $(-1)^{(i+n)m} xy$ for $x \in P_i$ and $y \in Q$. In particular, we identify $A[n] \otimes_A A[m]$ to $A[n + m]$ by the isomorphism $\beta_{A, A}^n \alpha_{A[n], A}^m$ that sends $a \otimes b$ to $(-1)^{mn} ab$ for $a, b \in A$.

Definition 3. Given an A^e -projective resolution (P, μ_P) of A , a chain map $\Delta_P : P \rightarrow P^{\otimes_{A^n}}$ is called a *diagonal n -approximation* of P if $\mu_P^{\otimes n} \Delta_P = \mu_P$.

Let (P, μ_P) be an A^e -projective resolution of A . Suppose also that $\Delta_P : P \rightarrow P \otimes_A P$ is a diagonal 2-approximation of P . Then the operation \smile_{Δ_P} on (P, A) defined for $f : P \rightarrow A[-n]$ and $g : P \rightarrow A[-m]$ by the equality $f \smile_{\Delta_P} g = (-1)^{mn}(f \otimes g)\Delta_P$ induces the cup product on $\mathrm{HH}^*(A)$. Note also that if $f \in C^n(A)$ and $g \in C^m(A)$, then the equality $f \smile g = (-1)^{mn}(f \otimes g)\Delta$ holds for Δ defined by the equality

$$\Delta(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) = \sum_{i=0}^n (1 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1) \otimes_A (1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes 1). \quad (3.1)$$

4 Gerstenhaber bracket via an arbitrary resolution

In this section we prove some new formulas for the Gerstenhaber bracket. The existence of these formulas is based on the following lemma.

Lemma 2. Let (P, μ_P) be an A^e -projective resolution of A and $f : P \rightarrow A[-n]$ be such that $fd_P = 0$. Then $f \otimes 1_P - 1_P \otimes f : P \otimes_A P \rightarrow P[-n]$ is homotopic to 0.

Proof. It is easy to check that $\mathbf{d}(f \otimes 1_P - 1_P \otimes f) = 0$. Since $\mu_P(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0$, there is some map $\phi : P \otimes_A P \rightarrow P[1]$ such that $\mu_P \otimes 1_P - 1_P \otimes \mu_P = \mathbf{d}\phi$. Then $\mu_P(f \otimes 1_P - 1_P \otimes f) = -f\mathbf{d}\phi \sim 0$ and so $f \otimes 1_P - 1_P \otimes f \sim 0$ by Lemma 1. □

Corollary 3. Let P, f be as above and Δ_P be some diagonal 2-approximation of P . Then $(f \otimes 1_P - 1_P \otimes f)\Delta_P : P \rightarrow P[-n]$ is homotopic to 0.

Proof. Since $\mathbf{d}\Delta_P = 0$, everything follows directly from Lemma 2. □

Definition 4. Let P, f and Δ_P be as above. We call $\phi_f : P \rightarrow P[1-n]$ a *homotopy lifting* of (f, Δ_P) if $\mathbf{d}\phi_f = (f \otimes 1_P - 1_P \otimes f)\Delta_P$ and $\mu_P\phi_f + f\phi \sim 0$ for some $\phi : P \rightarrow P[1]$ such that $\mathbf{d}\phi = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$.

Note that it follows from the proofs of Lemma 1 and Lemma 2 that some homotopy lifting exists for any cocycle. Now we are ready to prove our first formula.

Theorem 4. Let (P, μ_P) be an A^e -projective resolution of A and $\Delta_P : P \rightarrow P \otimes_A P$ be a diagonal 2-approximation of P . Let $f : P \rightarrow A[-n]$ and $g : P \rightarrow A[-m]$ represent some cocycles. Suppose that ϕ_f and ϕ_g are homotopy liftings for (f, Δ_P) and (g, Δ_P) respectively. Then the Lie bracket of the classes of f and g can be represented by the class of the element

$$[f, g]_{\phi, \Delta_P} = (-1)^m f\phi_g + (-1)^{m(n-1)} g\phi_f. \quad (4.1)$$

Proof. We will prove the assertion on the theorem in three steps.

1. Let us prove that the operation induced on the Hochschild cohomology by $[\cdot, \cdot]_{\phi, \Delta_P}$ does not depend on the choice of Δ_P and ϕ . We do this in two steps:

- If ϕ_g and ϕ'_g are two homotopy liftings for g , then $\mathbf{d}(\phi_g - \phi'_g) = 0$ and $\mu_P(\phi_g - \phi'_g) \sim g\epsilon$ for some chain map $\epsilon : P \rightarrow P[1]$. Then $\epsilon \sim 0$ and $\mu_P(\phi_g - \phi'_g) \sim 0$. Hence, $\phi_g - \phi'_g \sim 0$ and $f\phi'_g \sim f\phi_g$. Analogously, $g\phi'_f \sim g\phi_f$ and so $[f, g]_{\phi', \Delta} \sim [f, g]_{\phi, \Delta}$.
- Let Δ'_P and Δ_P be two diagonal 2-approximation of P and ϕ_f and ϕ_g be homotopy liftings for (f, Δ_P) and (g, Δ_P) correspondingly. Then $\Delta'_P = \Delta_P + \mathbf{d}u$ for some u . Hence, $\phi'_f = \phi_f + (f \otimes 1_P - 1_P \otimes f)u$ and $\phi'_g = \phi_g + (g \otimes 1_P - 1_P \otimes g)u$ are homotopy liftings for (f, Δ'_P) and (g, Δ'_P) . It is easy to see that $[f, g]_{\phi', \Delta'} = [f, g]_{\phi, \Delta}$.

2. Let us prove that the operation induced on the Hochschild cohomology does not depend on the choice of an A^e -projective resolution of A . Let (Q, μ_Q) be another A^e -projective resolution of A . Let $\Phi_P^Q : P \rightarrow Q$ and $\Phi_Q^P : Q \rightarrow P$ be comparison morphisms, and ϕ_f and ϕ_g be homotopy liftings for (f, Δ_P) and (g, Δ_P) correspondingly. There is some u such that $\mathbf{d}u = 1_P - \Phi_Q^P \Phi_P^Q$. Let us prove that

$$\Phi_P^Q(\phi_f - (fu \otimes 1_P - 1_P \otimes fu)\Delta_P)\Phi_Q^P \text{ and } \Phi_P^Q(\phi_g - (gu \otimes 1_P - 1_P \otimes gu)\Delta_P)\Phi_Q^P$$

are homotopy liftings for $(f\Phi_Q^P, \Delta_Q)$ and $(g\Phi_Q^P, \Delta_Q)$ correspondingly. Here Δ_Q denotes the map $(\Phi_P^Q \otimes \Phi_P^Q)\Delta_P \Phi_Q^P$. The first condition from the definition of the homotopy lifting can be verified by direct computations. Let us verify the second condition for

$$\psi = \Phi_P^Q \phi_f \Phi_Q^P - \Phi_P^Q(fu \otimes 1_P - 1_P \otimes fu)\Delta_P \Phi_Q^P.$$

We have $\mu_P \phi_f + f\epsilon \sim 0$ for some ϵ such that $\mathbf{d}\epsilon = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$. Now we have $\mu_Q \psi + f(\epsilon + (u \otimes \mu_P - \mu_P \otimes u)\Delta_P)\Phi_Q^P \sim 0$. One can check that

$$\mathbf{d}(\epsilon + (u \otimes \mu_P - \mu_P \otimes u)\Delta_P - \Phi_Q^P \Phi_P^Q \epsilon) = 0.$$

Hence, $\epsilon + (u \otimes \mu_P - \mu_P \otimes u)\Delta_P \sim \Phi_Q^P \Phi_P^Q \epsilon$ by Lemma 1 and $\mu_Q \psi + (f\Phi_Q^P)(\Phi_P^Q \epsilon \Phi_Q^P) \sim 0$. It remains to note that $\mathbf{d}(\Phi_P^Q \epsilon \Phi_Q^P) = (\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q$.

3. Suppose now that $(P, \mu_P) = (\text{Bar}(A), \mu_A)$ and $\Delta_P = \Delta$, where Δ is a map from (3.1). Let us define

$$\begin{aligned} & \phi_g(1 \otimes a_1 \otimes \cdots \otimes a_{m+i} \otimes 1) \\ &= \sum_{j=1}^i (-1)^{(m-1)j-1} \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes g(a_j \otimes \cdots \otimes a_{j+m-1}) \otimes a_{j+m} \otimes \cdots \otimes a_{i+m-1} \otimes 1 \end{aligned}$$

and analogously for ϕ_f . Then we have $(-1)^m f\phi_g + (-1)^{m(n-1)} g\phi_f = [f, g]$ by definition. Direct calculations show that ϕ_f and ϕ_g are homotopy liftings for (f, Δ) and (g, Δ) .

□

Let (P, μ_P) be an A^e -projective resolution for A , and $\Delta_P^{(2)} : P \rightarrow P \otimes_A P \otimes_A P$ be a diagonal 3-approximation of P . There is some homotopy ϕ_P for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. Since

$$(\mu_P \otimes \mu_P)(\mu_P \otimes 1_P \otimes 1_P - 1_P \otimes 1_P \otimes \mu_P)\Delta_P^{(2)} = 0,$$

there is some homotopy ϵ_P for $(\mu_P \otimes 1_P \otimes 1_P - 1_P \otimes 1_P \otimes \mu_P)\Delta_P^{(2)}$. We define

$$f \circ_{\Delta_P^{(2)}, \phi_P, \epsilon_P} g = f\phi_P(1_P \otimes g \otimes 1_P)\Delta_P^{(2)} - (-1)^m(f \otimes g)\epsilon_P : P \rightarrow A[1 - n - m] \quad (4.2)$$

and

$$[f, g]_{\Delta_P^{(2)}, \phi_P, \epsilon_P} = f \circ_{\Delta_P^{(2)}, \phi_P, \epsilon_P} g - (-1)^{(n-1)(m-1)} g \circ_{\Delta_P^{(2)}, \phi_P, \epsilon_P} f.$$

This formula is a slightly corrected variant of the formula from [4].

Corollary 5. *The operation $[\cdot, \cdot]_{\Delta_P^{(2)}, \phi_P, \epsilon_P}$ induces an operation on $\mathrm{HH}^*(A)$ that coincides with the usual Lie bracket on the Hochschild cohomology.*

Proof. By Theorem 4 it is enough to check that $-(1_P \otimes g)\epsilon_P + (-1)^m\phi_P(1_P \otimes g \otimes 1_P)\Delta_P^{(2)}$ is a homotopy lifting for $(g, (\mu_P \otimes 1_P \otimes 1_P)\Delta_P^{(2)})$ if $gd_P = 0$. The first condition can be verified by a direct computation. The second condition can be easily verified after noting that $\mu_P\phi_P = 0$. □

Remark 1. Usually the diagonal 3-approximation $\Delta_P^{(2)}$ is constructed using some 2-approximation Δ_P by the rule $\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P$. It often occurs that the maps Δ_P and μ_P satisfy the equality

$$(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P. \quad (4.3)$$

In this case some things becomes easier. Firstly, one can set $\phi = 0$ in the definition of homotopy lifting. Then the second condition simply means that $\mu_P\phi_f$ is a coboundary. In particular, one can simply set $\phi_f|_{P_{n-1}} = 0$. Secondly, if (4.3) holds and the diagonal 3-approximation is defined as above, then one can set $\epsilon_P = 0$ in equality (4.2).

On the other hand, we always can set $\epsilon_P = (\phi_P \otimes 1_P + 1_P \otimes \phi_P)\Delta_P^{(2)}$ and obtain the following formula for the bracket:

$$\begin{aligned} [f, g]_{\Delta_P^{(2)}, \phi_P, \epsilon_P} &= -f\phi_P(g \otimes 1_P \otimes 1_P - 1_P \otimes g \otimes 1_P + 1_P \otimes 1_P \otimes g)\Delta_P^{(2)} \\ &\quad + (-1)^{(n-1)(m-1)}g\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}. \end{aligned} \quad (4.4)$$

5 A formula via contracting homotopy

In this section we present a formula that expresses the Lie bracket on the Hochschild cohomology in terms of an arbitrary resolution and a left contracting homotopy for it.

Definition 5. Let (P, μ_P) be a projective A^e -resolution of A . Let $t_P : P \rightarrow P$ and $\eta_P : A \rightarrow P$ be homomorphisms of left modules such that $t_P(P_i) \subset P_{i+1}$ and $\eta_P(A) \subset P_0$. The pair (t_P, η_P) is called a *left contracting homotopy* for (P, μ_P) if $d_P t_P + t_P d_P + \eta_P \mu_P = 1_P$ and $t_P(t_P + \eta_P) = 0$.

Since A is projective as left A -module, any A^e -projective resolution of A splits as a complex of left A -modules. Hence, a left contracting homotopy exists for any A^e -projective resolution of A (see [2, Lemma 2.3] and the remark after it for details).

Let us fix an A^e -projective resolution (P, μ_P) of A and a left contracting homotopy (t_P, η_P) for it.

For any $n \geq 0$ the map $\pi_n : A \otimes P_n \rightarrow P_n$ defined by the equality $\pi_n(x \otimes a) = xa$ for $a \in A, x \in P_n$ is an epimorphism of A -bimodules. Since P_n is projective, there is $\iota_n \in \text{Hom}_{A^e}(P_n, A \otimes P_n)$ such that $\pi_n \iota_n = 1_{P_n}$. Let us fix such ι_n for each $n \geq 0$. Then π_n and ι_n ($n \geq 0$) determine homomorphisms of graded A -bimodules $\pi : A \otimes P \rightarrow P$ and $\iota : P \rightarrow A \otimes P$.

Let us define

$$\begin{aligned} t_L &:= (1_P \otimes \pi)(t_P \otimes 1_P)(1_P \otimes \iota) : P \otimes_A P \rightarrow (P \otimes_A P)[1], \\ \eta_L &:= (1_P \otimes \pi)(\eta_P \otimes 1_P)\iota : P \rightarrow P \otimes_A P, \\ d_L &:= d_P \otimes 1_P, d_R := 1_P \otimes d_P : P \otimes_A P \rightarrow (P \otimes_A P)[-1], \\ \mu_L &:= \mu_P \otimes 1_P, \mu_R := 1_P \otimes \mu_P : P \otimes_A P \rightarrow A. \end{aligned}$$

Note that all the defined maps are homomorphisms of A -bimodules. Note also that we omit isomorphisms $\alpha_{P,P}^1$ and $\beta_{P,P}^{\pm 1}$ in our definitions according to our agreement. It is easy to see that the map $t_L d_R : P \otimes_A P \rightarrow P \otimes_A P$ is locally nilpotent in the sense that for any $x \in P \otimes_A P$ there is an integer l such that $(t_L d_R)^l(x) = 0$. Hence, the map $1_{P \otimes_A P} + t_L d_R$ is invertible.

Let now $f : P \rightarrow A[-n]$ and $g : P \rightarrow A[-m]$ be maps of complexes. Let us define

$$f \circ g = -f \mu_R S t_L (1_P \otimes g \otimes 1_P) (1_P \otimes S \eta_L) S \eta_L,$$

where $S = (1_{P \otimes_A P} + t_L d_R)^{-1}$.

Theorem 6. *In the notation above the operation defined by the equality $[f, g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f$ induces the usual Lie bracket on the Hochschild cohomology.*

We divide the proof into several lemmas. First of all, note that

$$d_L t_L + t_L d_L + \eta_L \mu_L = 1_{P \otimes_A P}, \mu_L \eta_L = 1_P, (d_R)^2 = (d_L)^2 = 0 \text{ and } d_L d_R + d_R d_L = 0. \quad (5.1)$$

Lemma 7. $(d_L + d_R)S = S(d_L + \eta_L \mu_L d_R)$.

Proof. Let us multiply the desired equality by $1_{P \otimes_A P} + t_L d_R$ on the left and on the right at the same time. We obtain that we have to prove that

$$d_L + d_R + t_L d_R d_L + t_L (d_R)^2 = d_L + \eta_L \mu_L d_R + d_L t_L d_R + \eta_L \mu_L d_R t_L d_R.$$

Using (5.1) one can see that it is enough to show that $\eta_L \mu_L d_R t_L d_R = 0$. But the last equality follows from the fact that the image of $d_R t_L d_R$ lies in $\oplus_{n>0} P_n \otimes_A P \subset \text{Ker } \mu_L$.

□

Lemma 8. $S \eta_L$ is a diagonal 2-approximation of P .

Proof. By Lemma 7 we have

$$\mathbf{d}(S\eta_L) = (d_L + d_R)S\eta_L - S\eta_L d_P = S(d_L + \eta_L \mu_L d_R)\eta_L - S\eta_L d_P.$$

Since $\text{Im } \eta_L \subset \text{Ker } d_L$, it is enough to prove that $\eta_L \mu_L d_R \eta_L = \eta_L d_P$. It is easy to see that $\mu_L d_R = d_P \mu_L$. Hence, $\eta_L \mu_L d_R \eta_L = \eta_L d_P \mu_L \eta_L = \eta_L d_P$ by (5.1).

□

Proof of Theorem 6. It follows from Lemma 8 that $\Delta = (1_P \otimes \mu_R S\eta_L)S\eta_L$ is a diagonal 2-approximation of P .

It is enough to show that $\phi_g = (-1)^{m-1} \mu_R St_L(1_P \otimes g \otimes 1_P)(1_P \otimes S\eta_L)S\eta_L$ is a homotopy lifting for (g, Δ) .

Direct calculations show that $\mathbf{d}\phi_g = (g \otimes 1_P - 1_P \otimes g)\Delta$. Note also that $\mu_L \Delta = \mu_R S\eta_L = \mu_R \Delta$ and $\mu_P \phi_g = 0$. Hence, ϕ_g is a homotopy lifting for (g, Δ) and the theorem is proved.

□

6 Formulas for the Connes' differential

In this section we discuss some formulas for the Connes' differential. These formulas are based on the formula from [5]. In the case of a symmetric algebra a formula for the Connes' differential gives a formula for a BV differential. Thus, we obtain in this section an alternative way for computing the Lie bracket on the Hochschild cohomology of a symmetric algebra.

Let Tr denote the functor $A \otimes_{A^e} -$ from the category of A -bimodules to the category of \mathbf{k} -linear spaces. If M and N are A -bimodules, then there is an isomorphism $\sigma_{M,N} : Tr(M \otimes_A N) \rightarrow Tr(N \otimes_A M)$ defined by the equality $\sigma_{M,N}(1 \otimes x \otimes y) = 1 \otimes y \otimes x$ for $x \in M$ and $y \in N$. Moreover, for $f \in \text{Hom}_{A^e}(M_1, M_2)$ and $g \in \text{Hom}_{A^e}(N_1, N_2)$ one has $\sigma_{M_2, N_2} Tr(f \otimes g) = Tr(g \otimes f) \sigma_{M_1, N_1}$. It is easy to see also that Tr induces a functor from the category of A^e -complexes to the category of \mathbf{k} -complexes. In this case $\sigma_{P,Q}$ is defined by the equality $\sigma_{P,Q}(1 \otimes x \otimes y) = (-1)^{ij} \otimes y \otimes x$ for $x \in P_i$ and $y \in Q_j$ and satisfies the property $\sigma_{P_2, Q_2} Tr(f \otimes g) = Tr(g \otimes f) \sigma_{P_1, Q_1}$ for $f : P_1 \rightarrow P_2$ and $g : Q_1 \rightarrow Q_2$.

The Hochschild homology $\text{HH}_*(A)$ of the algebra A is simply the homology of the complex $Tr(\text{Bar}(A))$. As in the case of cohomology, any comparison morphism $\Phi_P^Q : P \rightarrow Q$ between resolutions (P, μ_P) and (Q, μ_Q) of the algebra A induces an isomorphism $Tr(\Phi_P^Q) : H_* Tr(P) \rightarrow H_* Tr(Q)$. Thus, the Hochschild homology of A is isomorphic to the homology of $Tr(P)$ for any projective bimodule resolution (P, μ_P) of A .

Note that $Tr(\text{Bar}_n(A)) \cong A^{\otimes(n+1)}$. Connes' differential $\mathcal{B} : \text{HH}_n(A) \rightarrow \text{HH}_{n+1}(A)$ is the map induced by the map from $Tr(\text{Bar}_n(A))$ to $Tr(\text{Bar}_{n+1}(A))$ that sends $a_0 \otimes a_1 \otimes \dots \otimes a_n \in A^{\otimes(n+1)}$ to

$$\sum_{i=0}^n (-1)^{in} 1 \otimes a_i \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1} + \sum_{i=0}^n (-1)^{in} a_i \otimes 1 \otimes a_{i+1} \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1}.$$

In fact, it follows from some standard arguments that the homological class of the second summand is zero. The following result is essentially stated in [5].

Proposition 9 (D. Kaledin). *Let (P, μ_P) be a projective bimodule resolution of A , Δ_P be a diagonal 2-approximation for P , and $\phi_P : P \otimes_A P \rightarrow P[1]$ be such that $\mu_P \otimes 1_P - 1_P \otimes \mu_P = \mathbf{d}\phi$. Then the map*

$$Tr(\phi_P)(1 + \sigma_{P,P})Tr(\Delta_P) : Tr(P) \rightarrow Tr(P[1])$$

induces the Connes's differential on the Hochschild homology.

This result can be written in a slightly different form.

Corollary 10. *Let (P, μ_P) , Δ_P , and ϕ_P be as in Proposition 9, and $\epsilon : P \rightarrow P[1]$ be such that $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P = \mathbf{d}\epsilon$. Then the map*

$$Tr(\phi_P)\sigma_{P,P}Tr(\Delta_P) + Tr(\epsilon) : Tr(P) \rightarrow Tr(P[1])$$

induces the Connes's differential on the Hochschild homology.

Proof. Since $\mathbf{d}(\phi\Delta_P) = (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$, it is enough to note that the map $H_*(Tr(\phi_P)\sigma_{P,P}Tr(\Delta_P) + Tr(\epsilon)) : HH_*(A) \rightarrow HH_*(A)$ does not depend on the choice of ϵ . □

Now it is not difficult to express the Connes' differential in terms of a contracting homotopy.

Corollary 11. *Let S , t_L and η_L be as in the previous section. Then the map $-Tr(\mu_R St_L)\sigma_{P,P}Tr((1_P \otimes (\mu_R S\eta_L)^2)S\eta_L)$ induces the Connes' differential on the Hochschild homology.*

Proof. It follows from Lemma 8 that $\mu_R S\eta_L : P \rightarrow P$ is a comparison morphism, i.e. there is some $u : P \rightarrow P[1]$ such that $1 - \mu_R S\eta_L = \mathbf{d}u$. It is not hard to show using Lemma 7 that $\mathbf{d}\phi_P = \mu_L - \mu_R$ for $\phi_P = u(\mu_L - \mu_R) - \mu_R St_L(\mu_R S\eta_L \otimes 1_P)$. Let also note that $(\mu_L - \mu_R)\Delta_P = 0$ for $\Delta_P = (1_P \otimes \mu_R S\eta_L)S\eta_L$. Then the Connes' differential is induced by the map

$$Tr(\phi_P)\sigma_{P,P}Tr(\Delta_P) = -Tr(\mu_R St_L)\sigma_{P,P}Tr((1_P \otimes (\mu_R S\eta_L)^2)S\eta_L).$$

□

Now we explain how one can obtain a formula for a BV differential on the Hochschild cohomology of a symmetric algebra in terms of an arbitrary resolution.

First of all, let us recall that there are well known maps $\mathbf{i}_f : HH_*(A) \rightarrow HH_*(A)$ for $f \in HH^*(A)$, whose definition can be found, for example, in [9]. These maps satisfy the condition $\mathbf{i}_f \mathbf{i}_g = \mathbf{i}_{f \smile g}$. We need also the fact that $\mathbf{i}_f|_{HH_n(A)} = 0$ for $n < |f|$ and that $\mathbf{i}_f|_{HH_{|f|}(A)}$ is the map induced by $Tr(\tilde{f}) : Tr(P_n) \rightarrow Tr(A) \cong HH_0(A)$, where $\tilde{f} \in \text{Hom}_{A^e}(P_n, A)$ represents f . After the correction of signs one obtains by [9, Lemma 15] that

$$\mathbf{i}_{[f,g]}(x) = (-1)^{(|f|+1)|g|}(-(-1)^{|f|+|g|})\mathbf{B}\mathbf{i}_{f \smile g}(x) + \mathbf{i}_f \mathbf{B}\mathbf{i}_g(x) - (-1)^{|f||g|}\mathbf{i}_g \mathbf{B}\mathbf{i}_f(x) - \mathbf{i}_{f \smile g} \mathbf{B}(x)$$

for all $f, g \in HH^*(A)$, $x \in HH_*(A)$. Considering $x \in HH_{|f|+|g|-1}(A)$ one now obtains

$$Tr([f, g]) = -(-1)^{(|f|+1)|g|}(Tr(f \smile g)\mathbf{B} - Tr(f)\mathbf{B}\mathbf{i}_g - (-1)^{|f||g|}Tr(g)\mathbf{B}\mathbf{i}_f). \quad (6.1)$$

Definition 6. A *Batalin–Vilkovisky algebra* (BV algebra for short) is a Gerstenhaber algebra $(R^\bullet, \smile, [\ , \])$ with an operator $\mathcal{D}: R^\bullet \rightarrow R^{\bullet-1}$ of degree -1 such that $\mathcal{D} \circ \mathcal{D} = 0$ and

$$[a, b] = -(-1)^{(|a|+1)|b|}(\mathcal{D}(a \smile b) - \mathcal{D}(a) \smile b - (-1)^{|a|}a \smile \mathcal{D}(b))$$

for homogeneous elements $a, b \in R^\bullet$.

Definition 7. The finite dimensional algebra A is called *symmetric* if $A \cong \text{Hom}_{\mathbf{k}}(A, k)$ as an A -bimodule.

Let A be symmetric. Let $\theta: A \rightarrow \mathbf{k}$ be an image of 1 under some bimodule isomorphism from A to $\text{Hom}_{\mathbf{k}}(A, \mathbf{k})$. Then it is easy to see that θ induces a map from $\text{Tr}(A)$ to \mathbf{k} . We denote this map by θ too. Note also that if $f \in \text{Hom}_{A^e}(M, A)$, then $\theta \text{Tr}(f) = 0$ iff $f = 0$.

Let $\mathcal{B}_P: \text{Tr}(P) \rightarrow \text{Tr}(P[1])$ be a map inducing the Connes' differential on the Hochschild homology. Then we can define $\mathcal{D}_P(f): P \rightarrow A[1-n]$ for $f: P \rightarrow A[-n]$ as the unique map such that $\theta \text{Tr}(\mathcal{D}_P(f)) = \theta \text{Tr}(f)\mathcal{B}_P$. Applying θ to the equality (6.1) with $\mathcal{B} = \mathcal{B}_P$ one obtains

$$\begin{aligned} \theta \text{Tr}([f, g]) &= -(-1)^{(|f|+1)|g|}(\theta \text{Tr} \mathcal{D}_P(f \smile g) - \theta \text{Tr} \mathcal{D}_P(f)\mathbf{i}_g - (-1)^{|f||g|}\theta \text{Tr} \mathcal{D}_P(g)\mathbf{i}_f) \\ &= -(-1)^{(|f|+1)|g|}\theta \text{Tr}(\mathcal{D}_P(f \smile g) - \mathcal{D}_P(f) \smile g - (-1)^{|f|}f \smile \mathcal{D}_P(g)), \end{aligned}$$

i.e. \mathcal{D}_P induces a BV differential on the Hochschild cohomology.

Remark 2. One can show that the BV differential from [8, Theorem 1] coincides with the differential \mathcal{D}_P defined here. Note also that if one knows the BV differential and the cup product, then it is easy to compute the Gerstenhaber bracket.

7 Example of an application

In this section we apply the results of the previous sections to describe the BV structure on the Hochschild cohomology of the family of algebras considered in [6] and [7]. During this section we fix some integer $k > 1$ and set $A = \mathbf{k}\langle x_0, x_1 \rangle / \langle x_0^2, x_1^2, (x_0x_1)^k - (x_1x_0)^k \rangle$. The index α in the notation x_α is always specified modulo 2. If a is an element of $\mathbf{k}\langle x_0, x_1 \rangle$, then we denote by a its class in A too.

Let G be a subset of $\mathbf{k}\langle x_0, x_1 \rangle$ formed by the elements $(x_0x_1)^{i+1}$, $x_1(x_0x_1)^i$, $(x_1x_0)^i$, and $x_0(x_1x_0)^i$ for $0 \leq i \leq k-1$. Note that the classes of the elements from G form a basis of A . Let G denote this basis too. Let l_v denote the length of $v \in G$. Note that the algebra A is symmetric with θ defined by the equalities $\theta((x_0x_1)^k) = 1$ and $\theta(v) = 0$ for $v \in G \setminus \{(x_0x_1)^k\}$. For $v \in G$, we introduce $v^* \in G$ as the unique element such that $\theta(vv^*) = 1$. Note that $\theta(vw) = 0$ for $w \in G \setminus \{v^*\}$. For $a = \sum_{v \in G} a_v v \in A$, where $a_v \in \mathbf{k}$ for $v \in G$, we define $a^* := \sum_{v \in G} a_v v^* \in A$. It is clear that $(a^*)^* = a$ for any $a \in A$. If $v, w \in G$, then $\frac{v}{w}$ denotes $(v^*w)^*$. If there is such $u \in G$ that $wu = v$, then this u is unique and $\frac{v}{w} = u$. If there is no such u , then $\frac{v}{w} = 0$. Note that $\frac{\frac{v}{x_\alpha}x_\beta}{x_\alpha}$ is equal to $\frac{v}{x_\alpha}$ if $\alpha = \beta$ and $v \in \{x_\alpha, 1^*\}$, and is equal to 0 in all remaining cases. For $a = \sum_{v \in G} a_v v \in A$ and $b = \sum_{v \in G} b_v v \in A$, where $a_v, b_v \in \mathbf{k}$ for $v \in G$, we define $\frac{a}{b} := \sum_{v, w \in G} a_v b_w \frac{v}{w} \in A$.

In this section we will use the bimodule resolution of A described in [6]. Here we present it in a little another form, but one can easily check that it is the same resolution. Let us introduce the algebra $B = \mathbf{k}[x_0, x_1, z]/\langle x_0 x_1 \rangle$. We introduce the grading on B by the equalities $|x_0| = |x_1| = 1$ and $|z| = 2$. Let us define the A^e -complex P . We set $P = A \otimes B \otimes A$ as an A -bimodule. The grading on P comes from the grading on B and the trivial grading on A . Let \bar{a} ($a \in B$) denote $1 \otimes a \otimes 1$. For convenience we set $\bar{a} = 0$ if $a = x_\alpha^i z^j$, where $\alpha \in \{0, 1\}$ and i or j is less than 0. We define the differential d_P by the equality

$$d_P(\overline{x_\alpha^i z^j}) = \begin{cases} 0, & \text{if } i = j = 0, \\ x_\alpha \overline{x_\alpha^{i-1}} + (-1)^i \overline{x_\alpha^{i-1}} x_\alpha, & \text{if } j = 0, i > 0, \\ \sum_{v \in G, \beta \in \{0, 1\}} (-1)^{j l_v + \beta} v^* \overline{x_\beta z^{j-1}} \frac{v}{x_\beta}, & \text{if } i = 0, j > 0, \\ x_\alpha \overline{x_\alpha^{i-1} z^j} + (-1)^{i+j} \overline{x_\alpha^{i-1} z^j} x_\alpha \\ + (-1)^{i+\alpha} ((-1)^j x_\alpha^* \overline{x_\alpha^{i+1} z^{j-1}} + \overline{x_\alpha^{i+1} z^{j-1}} x_\alpha^*), & \text{if } i, j > 0. \end{cases}$$

for $\alpha \in \{0, 1\}$, $i, j \geq 0$. We define $\mu_P : P_0 \rightarrow A$ by the equality $\mu_P(\bar{1}) = 1$. Then one can check that (P, μ_P) is an A^e -projective resolution of A isomorphic to the resolution from [6]. Let us define the left contracting homotopy (t_P, η_P) for (P, μ_P) . We define η_P by the equality $\eta_P(1) = \bar{1}$. Now, for $v \in G$, $\alpha \in \{0, 1\}$ and $i, j \geq 0$, we define

$$t_P(\overline{x_\alpha^i z^j} v) = \begin{cases} \sum_{w \in G, \beta \in \{0, 1\}} (-1)^{j(l_w + l_v + 1) + 1} \frac{w^* \overline{x_\beta z^j}}{v^* x_\beta z^j} \frac{w}{x_\beta}, & \text{if } i = 0, v \neq 1^*, \\ \sum_{w \in G} (-1)^{j(l_w + 1) + 1} w^* \overline{x_{l_w} z^j} \frac{w}{x_{l_w}}, & \text{if } i = 0 \text{ and } v = 1^*, \\ (-1)^{i+j+1} \overline{x_\alpha^{i+1} z^j} \frac{v}{x_\alpha} + (-1)^{j l_v + j + l_v} \frac{v}{x_1^* z^{j+1}}, & \text{if } i = 1 \text{ and } \alpha = 1, \\ (-1)^{i+j+1} \overline{x_\alpha^{i+1} z^j} \frac{v}{x_\alpha}, & \text{otherwise.} \end{cases}$$

In this section we will use the notation of Section 5. Our aim is to describe the BV structure on the Hochschild cohomology of A . As it was explained in the previous section, it is enough to describe the Connes' differential. By Corollary 11 we have to describe the map $-Tr(St_L)\sigma_{P,P}Tr((1_P \otimes (\mu_R S\eta_L)^2)S\eta_L)$. Let us start with the map $S\eta_L : P \rightarrow P \otimes_A P$.

Firstly, let introduce the following notation:

$$\begin{aligned} A_{t,j} &= \sum_{\substack{v, w \in G \\ \alpha, \beta \in \{0, 1\}}} (-1)^{j l_v + t(l_w + l_v + 1) + \beta} \frac{w^* \overline{x_\alpha z^t}}{v} \frac{w}{x_\alpha} \otimes \overline{x_\beta z^{j-1}} \frac{v}{x_\beta}, \\ B_{t,j} &= \sum_{v \in G, \beta \in \{0, 1\}} (-1)^{(j+t)(l_v + 1)} \frac{x_{\beta+1}^* \overline{x_{\beta+1}^2 z^t} (x_\beta x_{\beta+1})^{k-1}}{v} \otimes \overline{x_\beta^2 z^{j-2}} \frac{v}{x_\beta}, \\ C_{t,i,j,\alpha} &= (-1)^{i+j+\alpha} \sum_{w \in G, \beta \in \{0, 1\}} (-1)^{t l_w} \frac{w^* \overline{x_\beta z^t}}{x_\alpha} \frac{w}{x_\beta} \otimes \overline{x_\alpha^{i+1} z^{j-1}}, \\ D_{t,i,j,\alpha} &= (-1)^{(i+1)t} \sum_{v \in G, \beta \in \{0, 1\}} (-1)^{j l_v + \beta} \frac{x_\alpha^{i+1} \overline{z^t} v^*}{x_\alpha} \otimes \overline{x_\beta z^{j-1}} \frac{v}{x_\beta}, \\ E_{t,i,j,\alpha} &= (x_{\alpha+1} x_\alpha)^{k-1} \overline{x_{\alpha+1}^2 z^t} (x_\alpha x_{\alpha+1})^{k-1} \otimes \overline{x_\alpha^{i+2} z^{j-2}} \\ &\quad + (-1)^{it} \overline{x_\alpha^{i+2} z^t} (x_{\alpha+1} x_\alpha)^{k-1} \otimes \overline{x_{\alpha+1}^2 z^{j-2}} (x_\alpha x_{\alpha+1})^{k-1}. \end{aligned}$$

Lemma 12. *If $q, j \geq 0$ are some integers, then*

$$S(\overline{z^q} \otimes \overline{z^j}) = \sum_{t=0}^j (-1)^{(j+q+1)t} (\overline{z^{q+t}} \otimes \overline{z^{j-t}} + A_{q+t,j-t} + B_{q+t,j-t}).$$

$$\text{In particular, } S\eta_L(\overline{z^j}) = \sum_{t=0}^j (-1)^{(j+1)t} (\overline{z^t} \otimes \overline{z^{j-t}} + A_{t,j-t} + B_{t,j-t}).$$

Proof. We have to show that

$$\overline{z^q} \otimes \overline{z^j} = \sum_{t=0}^j (-1)^{(j+q+1)t} (1_{P \otimes_A P} + t_L d_R) (\overline{z^{q+t}} \otimes \overline{z^{j-t}} + A_{q+t,j-t} + B_{q+t,j-t}). \quad (7.1)$$

Direct calculations show that $t_L d_R(\overline{z^{q+t}} \otimes \overline{z^{j-t}}) + A_{q+t,j-t} = 0$ and $t_L d_R B_{q+t,j-t} = 0$ for $0 \leq t \leq j$. One can show that if $t_P(\overline{x_\alpha z^{q+t}} \frac{w}{x_\alpha} x_\beta) \neq 0$, then either $\frac{w^*}{x_\beta} = 0$ or $w = x_0^*$, $\alpha = 1$ and $\beta = 0$. In the first case $\frac{w^*}{v} = 0$ or $\frac{v}{x_\beta} = 0$ for any $v \in G$. In the second case we have $t_P(\overline{x_1 z^{q+t}} \frac{x_0^*}{x_1} x_0) = -\overline{z^{q+t+1}}$, and $\frac{x_0}{v} \neq 0$ and $\frac{v}{x_0} \neq 0$ simultaneously only for $v = x_0$. Analogously, we have $t_P(\overline{x_\alpha z^{q+t}} \frac{w}{x_\alpha} x_\beta^*) \neq 0$ only if either $\frac{w^*}{x_\beta} = 0$ or $w = x_\alpha$, $\alpha = \beta + 1$. In the last case $t_P(\overline{x_\alpha z^{q+t}} x_\beta^*) = (-1)^{q+t} \overline{x_{\beta+1}^2 z^{q+t}} (x_\beta x_{\beta+1})^{k-1}$. Thus,

$$t_L d_R A_{q+t,j-t} + B_{q+t,j-t} + (-1)^{j+q+1} \overline{z^{q+t+1}} \otimes \overline{z^{j-t-1}} = 0.$$

Substituting the obtained values of $t_L d_R(\overline{z^{q+t}} \otimes \overline{z^{j-t}})$, $t_L d_R A_{q+t,j-t}$, and $t_L d_R B_{q+t,j-t}$ to (7.1) we obtain a true equality. □

Lemma 13. *If $\alpha \in \{0, 1\}$, and $q, j \geq 0$ and $i > 0$ are some integers, then*

$$S(\overline{z^q} \otimes \overline{x_\alpha^i z^j}) = \sum_{t=0}^j (-1)^{(i+j+q+1)t} \left(\sum_{r=0}^i (-1)^{r(q+t)} \overline{x_\alpha^r z^{q+t}} \otimes \overline{x_\alpha^{i-r} z^{j-t}} \right. \\ \left. + C_{q+t,i,j-t,\alpha} + D_{q+t,i,j-t,\alpha} + E_{q+t,i,j-t,\alpha} \right).$$

In particular,

$$S\eta_L(\overline{x_\alpha^i z^j}) = \sum_{t=0}^j (-1)^{(i+j+1)t} \left(\sum_{r=0}^i (-1)^{rt} \overline{x_\alpha^r z^t} \otimes \overline{x_\alpha^{i-r} z^{j-t}} + C_{t,i,j-t,\alpha} + D_{t,i,j-t,\alpha} + E_{t,i,j-t,\alpha} \right).$$

Proof. Firstly, note that $t_P(\overline{x_\alpha^r z^{q+t}} x_\alpha) = (-1)^{r+q+t+1} \overline{x_\alpha^{r+1} z^{q+t}}$. Also $t_P(\overline{x_1 z^{q+t}} x_1^*) = -\overline{z^{q+t+1}}$ and $t_P(\overline{x_\alpha^r z^{q+t}} x_\alpha^*) = 0$ for $r > 1$ and for $r = 1$, $\alpha = 0$. Hence, we have

$$(1_{P \otimes_A P} + t_L d_R) \left(\sum_{r=0}^i (-1)^{r(q+t)} \overline{x_\alpha^r z^{q+t}} \otimes \overline{x_\alpha^{i-r} z^{j-t}} \right) \\ = \overline{z^{q+t}} \otimes \overline{x_\alpha^i z^{j-t}} - C_{q+t,i,j-t,\alpha} - D_{q+t,i,j-t,\alpha} + (-1)^{i+j+q} \overline{\alpha z^{q+t+1}} \otimes \overline{x_\alpha^i z^{j-t-1}},$$

Now we have $t_P(\overline{x_\beta z^{q+t} \frac{w}{x_\beta} x_\alpha^*}) \neq 0$ only if either $w = x_\beta$ or $w = x_{\alpha+1} x_\alpha$, $\beta = \alpha + 1$. In the last case and in the case $w = x_\beta$, $\beta = \alpha$ we have $\frac{w^*}{x_\alpha} = 0$. Thus, the only nonzero case is $t_P(\overline{x_{\alpha+1} z^{q+t} x_\alpha^*}) = (-1)^{q+t} \overline{x_{\alpha+1} z^{q+t}} (x_\alpha x_{\alpha+1})^{k-1}$. Further, we have $t_P(\overline{x_\beta z^{q+t} \frac{w}{x_\beta} x_\alpha}) \neq 0$ only if either $\frac{w^*}{x_\alpha} = 0$ or $w = x_0^*$, $\alpha = 0$ and $\beta = 1$. In the last case we have $t_P(\overline{x_1 z^{q+t} x_1^*}) = -\overline{z^{q+t+1}}$. Thus,

$$\begin{aligned} t_L d_R C_{q+t, i, j-t, \alpha} &= -(x_{\alpha+1} x_\alpha)^{k-1} \overline{x_{\alpha+1}^2 z^{q+t}} (x_\alpha x_{\alpha+1})^{k-1} \otimes \overline{x_\alpha^{i+2} z^{j-t-2}} + (-1)^{i+j+q} (1-\alpha) \overline{z^{q+t+1}} \otimes \overline{x_\alpha^i z^{j-t-1}}. \end{aligned}$$

One can check that $t_P(\overline{x_\alpha^{i+1} z^{q+t} \frac{v^*}{x_\alpha} x_\beta}) = 0$ if $\frac{v}{x_\beta} \neq 0$. If $\frac{v^*}{x_\alpha} x_\beta^* \neq 0$, then either $v = x_\alpha^*$ or $v = (x_\alpha x_{\alpha+1})^*$, $\beta = \alpha + 1$. Since in the second case $\frac{v}{x_\beta} = 0$, we have

$$\begin{aligned} t_L d_R D_{q+t, i, j-t, \alpha} &= (-1)^{(i+1)(q+t+1)} t_L \left(\sum_{\beta \in \{0,1\}} \frac{\overline{x_\alpha^{i+1} z^{q+t} x_\beta^*} \otimes \overline{x_\beta^2 z^{j-t-2} x_\alpha^*}}{x_\beta} \right) \\ &= (-1)^{i(q+t)+1} \overline{x_\alpha^{i+2} z^{q+t}} (x_{\alpha+1} x_\alpha)^{k-1} \otimes \overline{x_{\alpha+1}^2 z^{j-t-2}} (x_\alpha x_{\alpha+1})^{k-1}. \end{aligned}$$

Finally, note that $t_L d_R E_{q+t, i, j-t, \alpha} = 0$. Taking in account all the proved equalities, we obtain

$$\begin{aligned} \sum_{t=0}^j (-1)^{(i+j+q+1)t} (1_{P \otimes_A P} + t_L d_R) \left(\sum_{r=0}^i (-1)^{rt} \overline{x_\alpha^r z^{q+t}} \otimes \overline{x_\alpha^{i-r} z^{j-t}} + C_{q+t, i, j-t, \alpha} + D_{q+t, i, j-t, \alpha} \right. \\ \left. + E_{q+t, i, j-t, \alpha} \right) = \sum_{t=0}^j (-1)^{(i+j+q+1)t} (\overline{z^{q+t}} \otimes \overline{x_\alpha^i z^{j-t}} + (-1)^{i+j+q} \overline{z^{q+t+1}} \otimes \overline{x_\alpha^i z^{j-t-1}}) = \overline{z^q} \otimes \overline{x_\alpha^i z^j}. \end{aligned}$$

□

From Lemmas 12 and 13 we obtain the following statement.

Lemma 14. $\mu_R S(\overline{z^q} \otimes \overline{x_\alpha^r z^t}) = (-1)^{q(r+t)} \overline{x_\alpha^{r+q+t}}$. In particular, $\mu_R S \eta_L = 1_P$.

It remains to describe $Tr(\mu_R S t_L) \sigma_{P,P}$ on the image of $Tr(S \eta_L)$.

Lemma 15. Let $v \in G$, $\alpha, \beta \in \{0, 1\}$, p, r, q and t be some integers. Suppose that $p > 0$ and one of the conditions $p = 1$, $\alpha = 1$, and $v \in \{x_1^*, 1^*\}$ is not satisfied. Then

$$\begin{aligned} \mu_R S t_L(\overline{x_\alpha^p z^q v} \otimes \overline{x_\beta^r z^t}) &= \begin{cases} (-1)^{(r+1)q+p+1} \overline{x_\alpha^{p+r+1} z^q} \frac{v}{x_\alpha}, & \text{if } t = 0 \text{ and either } v \in \{x_\alpha, 1^*\}, \beta = \alpha \text{ or } r = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Assume firstly that $\beta \neq \alpha$ and $r > 0$. Direct calculations show that $(t_L d_R)^2 t_L(\overline{x_\alpha^p z^q v} \otimes \overline{x_\beta^r z^t}) = 0$. Then

$$\mu_R S t_L(\overline{x_\alpha^p z^q v} \otimes \overline{x_\beta^r z^t}) = \mu_R (t_L - t_L d_R t_L + S(t_L d_R)^2 t_L)(\overline{x_\alpha^p z^q v} \otimes \overline{x_\beta^r z^t}) = 0.$$

Here we also use the fact that $t_L d_R t_L(\overline{x_\alpha^p z^q v} \otimes \overline{x_\beta}) = 0$ for the case $r = 1, t = 0$. In the remaining part of the proof we assume that $\beta = \alpha$.

Let us consider the case where $r > 0$. Direct calculations show that $t_L d_R t_L (\overline{x_\alpha^p z^q v} \otimes \overline{x_\alpha^r z^t}) = 0$ if $v \in G \setminus \{x_\alpha, 1^*\}$. One can also check that $t_L d_R t_L (\overline{x_\alpha^p z^q v} \otimes \overline{x_\alpha^r z^t}) = (-1)^q t_L (\overline{x_\alpha^{p+1} z^q v} \otimes \overline{x_\alpha^{r-1} z^t})$ if $v \in \{x_\alpha, 1^*\}$. Then

$$\begin{aligned} \mu_R St_L (\overline{x_\alpha^p z^q v} \otimes \overline{x_\alpha^r z^t}) &= \mu_R (t_L - St_L d_R t_L) (\overline{x_\alpha^p z^q v} \otimes \overline{x_\alpha^r z^t}) \\ &= \begin{cases} (-1)^{q+1} \mu_R St_L (\overline{x_\alpha^{p+1} z^q v} \otimes \overline{x_\alpha^{r-1} z^t}), & \text{if } v \in \{x_\alpha, 1^*\}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

If $v \in G \setminus \{x_\alpha, 1^*\}$, then the required equality is proved. If $v \in \{x_\alpha, 1^*\}$, then we obtain $\mu_R St_L (\overline{x_\alpha^p z^q v} \otimes \overline{x_\alpha^r z^t}) = (-1)^{r(q+1)} \mu_R St_L (\overline{x_\alpha^{p+r} z^q v} \otimes \overline{z^t})$ by induction. Hence, it remains to prove the required equality for $r = 0$. If $t = 0$, then everything is clear. If $t > 0$, then we have

$$\begin{aligned} \mu_R St_L (\overline{x_\alpha^p z^q v} \otimes \overline{z^t}) &= \mu_R (t_L - St_L d_R t_L) (\overline{x_\alpha^p z^q v} \otimes \overline{z^t}) \\ &= (-1)^{q+1} \mu_R St_L \left(\sum_{w \in G, \gamma \in \{0,1\}} (-1)^{j_{lw} + \gamma} \overline{x_\alpha^{p+1} z^q} \frac{v}{x_\alpha} w^* \overline{x_\gamma z^{t-1}} \frac{w}{x_\gamma} \right). \end{aligned}$$

It is easy to see that $v = w^* = x_\alpha$ if $\frac{v}{x_\alpha} w^* = x_\alpha$, and $w = \frac{v}{x_\alpha}$ if $\frac{v}{x_\alpha} w^* = 1^*$. Since in both cases $\frac{w}{x_\alpha} = 0$, we are done by the already proved equalities. \square

Lemma 16. 1. $\mu_R St_L (\overline{x_1 z^q x_1^*} \otimes \overline{x_\alpha^r z^t}) = (-1)^{(q+1)(r+t)+1} \overline{x_\alpha^r z^{q+t+1}}$ for $\alpha \in \{0, 1\}$, $q, r, t \geq 0$.
2. For $\alpha \in \{0, 1\}$ and integers r and t one has

$$\mu_R St_L (\overline{x_1 z^q 1^*} \otimes \overline{x_\alpha^r z^t}) = \begin{cases} (-1)^{(q+1)(r+t)+q} x_1 \overline{x_\alpha^r z^{q+1}} + (-1)^{(r+1)q} \overline{x_1^{r+2} z^q} x_1^*, & \text{if } t = 0 \text{ and } \alpha = 1, \\ (-1)^{(q+1)(r+t)+q} x_1 \overline{x_\alpha^r z^{q+t+1}}, & \text{otherwise.} \end{cases}$$

Proof. 1. Follows directly from Corollary 14.

2. Can be proved analogously to Lemma 15 using Corollary 14. \square

Lemma 17. If $v \in G$, $\alpha \in \{0, 1\}$, $i > 0$ and $j \geq t$, then

$$\begin{aligned} Tr(\mu_R St_L) \sigma_{P,P}(v \otimes B_{t,j-t}) &= Tr(\mu_R St_L) \sigma_{P,P}(v \otimes C_{t,i,j-t,\alpha}) \\ &= Tr(\mu_R St_L) \sigma_{P,P}(v \otimes E_{t,i,j-t,\alpha}) = 0. \end{aligned}$$

Proof. The equality $\mu_R St_L \sigma_{P,P}(v \otimes B_{t,j-t}) = \mu_R St_L \sigma_{P,P}(v \otimes E_{t,i,j-t,\alpha}) = 0$ follows directly from Lemma 15. Let now prove that $\mu_R St_L \sigma_{P,P}(v \otimes C_{t,i,j-t,\alpha}) = 0$. If $t > 0$, then the required equality follows directly from Lemma 15 again. Suppose that $t = 0$. Then we have

$$Tr(\mu_R St_L) \sigma_{P,P}(v \otimes C_{t,i,j-t,\alpha}) = (-1)^{j+\alpha} \otimes \sum_{w \in G, \beta \in \{0,1\}} \mu_R St_L \left(\overline{x_\alpha^{i+1} z^{j-1}} v \frac{w^*}{x_\alpha} \otimes \overline{x_\beta} \frac{w}{x_\beta} \right).$$

By Lemma 15 the expression $\mu_R St_L \left(\overline{x_\alpha^{i+1} z^{j-1}} v \frac{w^*}{x_\alpha} \otimes \overline{x_\beta} \frac{w}{x_\beta} \right)$ can be nonzero only in the case where $\alpha = \beta$ and $v \frac{w^*}{x_\alpha} \in \{x_\alpha, 1^*\}$. One can show that $v \frac{w^*}{x_\alpha} = x_\alpha$ only for $v = w^* = x_\alpha$. Since

in this case $\frac{w}{x_\alpha} = 0$, it remains to consider the case $v \frac{w^*}{x_\alpha} = 1^*$. In this case we have $v = wx_\alpha$ and so $w = \frac{x_\alpha^*}{v^*}$. By Lemma 15 we have

$$\mu_R St_L \left(\overline{x_\alpha^{i+1} z^{j-1}} 1^* \otimes \overline{x_\alpha} \frac{x_\alpha^*}{v^* x_\alpha} \right) = (-1)^{i \overline{x_\alpha^{i+3} z^{j-1}}} x_\alpha^* \frac{x_\alpha^*}{v^* x_\alpha}.$$

Since $\frac{x_\alpha^*}{v^* x_\alpha} \notin \{1, x_\alpha\}$, we have $\mu_R St_L \left(\overline{x_\alpha^{i+1} z^{j-1}} v \frac{w^*}{x_\alpha} \otimes \overline{x_\beta} \frac{w}{x_\beta} \right) = 0$ for all $w \in G$ and $\beta \in \{0, 1\}$. Hence, we are done. \square

Lemma 18. *If $v \in G$, $\alpha \in \{0, 1\}$, $i > 0$ and $j > t$, then*

$$\begin{aligned} Tr(\mu_R St_L) \sigma_{P,P}(v \otimes D_{t,i,j-t,\alpha}) \\ = \begin{cases} 0, & \text{if } \alpha = 1 \text{ and } v \in \{x_1, 1^*\}; \\ (-1)^{(j+1)(i+t+1)+(j-t)l_v} \frac{v}{x_\alpha} \otimes \overline{x_\alpha^{i+1} z^j}, & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Let us introduce the notation $a_{w,\beta} := \mu_R St_L \left(\overline{x_\beta z^{j-t-1}} \frac{w}{x_\beta} v \otimes \overline{x_\alpha^{i+1} z^t} \frac{w^*}{x_\alpha} \right)$. Then

$$Tr(\mu_R St_L) \sigma_{P,P}(v \otimes D_{t,i,j-t,\alpha}) = (-1)^{(i+1)(t+1)} \otimes \sum_{w \in G, \beta \in \{0,1\}} (-1)^{(j-t)l_w + \beta} a_{w,\beta}.$$

It follows from Lemmas 15 and 16 that if $a_{w,\beta} \neq 0$, then $\frac{w}{x_\beta} v \in \{x_\beta, x_1^*, 1^*\}$. Let us consider each of the mentioned values separately.

1. If $\frac{w}{x_\beta} v = x_\beta$, then one can show that $v = w = x_\beta$. Since in this case $\frac{w^*}{x_\beta} = 0$ and $a_{w,\beta}$ can be nonzero only for $\beta = \alpha$, we obtain $a_{w,\beta} = 0$.
2. If $\frac{w}{x_\beta} v = x_1^*$, then $a_{w,\beta}$ can be nonzero only for $\beta = 1$. One can show that $w = v^*$ in this case. On the other hand, $a_{v^*,1} = (-1)^{(j-t)(i+t+1)+1} \overline{x_\alpha^{i+1} z^j} \frac{v}{x_\alpha}$ if $vx_1 \neq 0$, and $a_{v^*,1} = 0$ if $vx_1 = 0$.
3. If $\frac{w}{x_\beta} v = 1^*$, then one can show that $w = x_\beta v^*$. One can show that $x_\beta^* \frac{x_\beta^*}{v^* x_\beta} = 0$. Then it follows from Lemmas 15 and 16 that

$$a_{x_\beta v^*, \beta} = \begin{cases} (-1)^{(j-t)(i+t+1)+1} x_1 \overline{x_\alpha^{i+1} z^j} \frac{x_1^*}{v^* x_\alpha}, & \text{if } \beta = 1; \\ 0, & \text{if } \beta = 0. \end{cases}$$

Note that if $vx_1 \neq 0$, then $\frac{x_1^*}{v^* x_\alpha} = 0$ and $a_{x_\beta v^*, 1} = 0$. If $vx_1 = 0$, then one can show that $\frac{x_1^*}{v^* x_\alpha} x_1 = \frac{v}{x_\alpha}$ except the case where $\alpha = 1$ and $v \in \{x_1, 1^*\}$.

Putting all the obtained equalities together we obtain the statement of the lemma. \square

Lemma 19. *If $v \in G$, $\alpha \in \{0, 1\}$, and $i > 0$, j and t are some integers, then*

$$\begin{aligned} Tr(\mu_R St_L) \sigma_{P,P} \left(\sum_{r=0}^i (-1)^{rt} \overline{x_\alpha^r z^t} \otimes \overline{x_\alpha^{i-r} z^{j-t}} \right) \\ = (-1)^{(j+\alpha+1)(i+\alpha+1)+(j+\alpha)l_v+t(j+l_v+1)} \frac{v}{x_\alpha^*} \otimes \overline{x_\alpha^{i-1} z^{j+1}} + V_{i,j,t,\alpha,v} \otimes \overline{x_\alpha^{i+1} z^j}, \end{aligned}$$

where

$$V_{i,j,t,\alpha,v} = \begin{cases} 0, & \text{if } t > 0, \alpha = 0 \text{ and } v \in \{x_0, 1^*\}; \\ (-1)^{j(i+l_v)+1+t(j+l_v+1)} \frac{x_\alpha^*}{v^*}, & \text{if } t > 0 \text{ and either } \alpha = 1 \text{ or } v \notin \{x_0, 1^*\}; \\ (-1)^{i+j+1} \frac{v}{x_\alpha} + (-1)^{j(i+l_v)+1} \frac{x_\alpha^*}{v^*}, & \text{if } t = 0 \text{ and } v \notin \{x_\alpha, 1^*\}; \\ \sum_{r=1}^{i+1} (-1)^{r(i+j)+1} \frac{v}{x_\alpha}, & \text{if } t = 0 \text{ and either } v = x_\alpha \text{ or } \alpha = 0, v = 1^*; \\ \left(\sum_{r=1}^i (-1)^{r(i+j)+1} + (-1)^{ij+1} \right) \frac{v}{x_\alpha}, & \text{if } t = 0, \alpha = 1 \text{ and } v = 1^*. \end{cases}$$

Proof. Using Lemmas 15 and 16 one can show that if $r < i$, then

$$\text{Tr}(\mu_R St_L) \sigma_{P,P}(v \otimes \overline{x_\alpha^r z^t} \otimes \overline{x_\alpha^{i-r} z^{j-t}}) = X_{i,r,j,t,\alpha,v} + Y_{i,r,j,t,\alpha,v},$$

where

$$X_{i,r,j,t,\alpha,v} = \begin{cases} (-1)^{(r+1)(i+j)+1} \frac{v}{x_\alpha} \otimes \overline{x_\alpha^{i+1} z^j}, & \text{if } t = 0 \text{ and either } v \in \{x_\alpha, 1^*\} \text{ or } r = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Y_{i,r,j,t,\alpha,v} = \begin{cases} (-1)^{ji+(j+1)l_v+t(i+j+l_v)} \frac{v}{x_1^*} \otimes \overline{x_1^{i-1} z^{j+1}}, & \text{if } r = i-1 \text{ and } \alpha = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let now calculate

$$\begin{aligned} \text{Tr}(\mu_R St_L) \sigma_{P,P}(v \otimes \overline{x_\alpha^i z^t} \otimes \overline{z^{j-t}}) &= 1 \otimes \mu_R(t_L - \mu_R St_L d_R t_L)(\overline{z^{j-t} v} \otimes \overline{x_\alpha^i z^t}) \\ &= 1 \otimes \mu_R St_L(t_P(\overline{z^{j-t} v}) \otimes (x_\alpha \overline{x_\alpha^{i-1} z^t} + (-1)^{i+t+\alpha} x_\alpha^* \overline{x_\alpha^{i+1} z^{t-1}})). \end{aligned}$$

One can show using Lemmas 15 and 16 that

$$\mu_R St_L(t_P(\overline{z^{j-t} v}) x_\alpha \otimes \overline{x_\alpha^{i-1} z^t}) = (-1)^{(j+1)(i+1)+jl_v+t(i+j+l_v+1)} (1-\alpha) \frac{v}{x_\alpha^*} \overline{x_\alpha^{i-1} z^{j+1}} + Z_{i,r,j,t,\alpha,v},$$

where

$$Z_{i,r,j,t,\alpha,v} = \begin{cases} (-1)^{j(i+l_v)+1} \frac{x_\alpha^*}{v^*} \overline{x_\alpha^{i+1} z^j}, & \text{if } t = 0 \text{ and either } \alpha = 1 \text{ or } v \neq 1^*; \\ (-1)^{j(i+1)+1} \frac{x_\alpha^{i+1} z^j}{x_0^*} x_0^*, & \text{if } t = 0, \alpha = 0 \text{ and } v = 1^*; \\ 0, & \text{otherwise.} \end{cases}$$

If $t > 0$, then using the same lemmas one can also show that

$$\mu_R St_L(t_P(\overline{z^{j-t} v}) x_\alpha^* \otimes \overline{x_\alpha^{i+1} z^{t-1}}) = (-1)^{(i+1)(t+1)+\alpha+1} V_{i,j,t,\alpha,v} \otimes \overline{x_\alpha^{i+1} z^j}.$$

Putting all the obtained equalities together, we obtain the statement of the lemma. \square

Lemma 20. *If $v \in G$, $\alpha \in \{0, 1\}$, and $t < j$ are integers, then*

$$\begin{aligned} & Tr(\mu_R St_L) \sigma_{P,P}(v \otimes A_{t,j-t}) \\ &= \begin{cases} 0, & \text{if } v \in \{1, x_1\}; \\ (-1)^{j+t+1} (x_1 x_0)^i \otimes \overline{x_0 z^j}, & \text{if } v = x_0 (x_1 x_0)^i, 0 \leq i \leq k-1; \\ ((-1)^{j+t+1} + (-1)^{j(t+1)+1}) (x_0 x_1)^i \otimes \overline{x_0 z^j}, & \text{if } v = x_1 (x_0 x_1)^i, 1 \leq i \leq k-1; \\ (-1)^{(j+1)(t+1)} (k-i+1) x_1 (x_0 x_1)^{i-1} \otimes \overline{x_0 z^j} \\ + (-1)^{(j+1)t+1} (k-i) x_0 (x_1 x_0)^{i-1} \otimes \overline{x_1 z^j}, & \text{if } v = (x_0 x_1)^i, 1 \leq i \leq k; \\ (-1)^{(j+1)(t+1)} (k-i) x_0 (x_1 x_0)^{i-1} \otimes \overline{x_1 z^j} \\ + (-1)^{(j+1)t+1} (k-i) x_1 (x_0 x_1)^{i-1} \otimes \overline{x_0 z^j}, & \text{if } v = (x_1 x_0)^i, 1 \leq i \leq k-1. \end{cases} \end{aligned}$$

Proof. Let us introduce the notation $a_{u,w,\alpha,\beta} := \mu_R St_L(\overline{x_\beta z^{j-t-1}} \frac{u}{x_\beta} v \frac{w^*}{u} \otimes \overline{x_\alpha z^t} \frac{w}{x_\alpha})$. Then

$$Tr(\mu_R St_L) \sigma_{P,P}(v \otimes A_{t,j-t}) = -1 \otimes \sum_{\substack{u, w \in G, \\ \alpha, \beta \in \{0, 1\}}} (-1)^{j l_u + t(l_w + 1) + \beta} a_{u,w,\alpha,\beta}.$$

It follows from Lemmas 15 and 16 that if $a_{u,w,\alpha,\beta} \neq 0$, then $\frac{u}{x_\beta} v \frac{w^*}{u} \in \{x_\beta, x_1^*, 1^*\}$. Let us consider each of the mentioned values separately.

1. If $\frac{u}{x_\beta} v \frac{w^*}{u} = x_\beta$, then one can show that $v = u = w^* = x_\beta$. Since in this case $\frac{w}{x_\beta} = 0$ and $a_{u,w,\alpha,\beta}$ can be nonzero only for $\beta = \alpha$, we obtain $a_{u,w,\alpha,\beta} = 0$.

2. If $\frac{u}{x_\beta} v \frac{w^*}{u} = x_1^*$, then $a_{u,w,\alpha,\beta}$ can be nonzero only for $\beta = 1$. Then we have $uv = wu \neq 0$. Since $\frac{u}{x_1} \neq 0$, we have $wx_1 \neq 0$ and $\frac{wu}{x_1} \neq 0$. Suppose that $2 \nmid l_v = l_w$. Then $w = x_0 (x_1 x_0)^i$ for some $0 \leq i \leq k-1$ and $u = w^* = v^*$. In this case we have $a_{v^*,v,0,1} = (-1)^{j(t+1)+1} \overline{x_0 z^j} \frac{v}{x_0}$, and $a_{v^*,v,1,1} = 0$.

Let now $2 \mid l_v = l_w$. Then $w = (x_1 x_0)^i$ for some $1 \leq i \leq k-1$ and $\frac{w}{x_\alpha} = 0$ for $\alpha = 0$. If $v = (x_1 x_0)^i$, then $u = (x_1 x_0)^j$ for some $1 \leq j \leq k-i$ and we have $a_{u,w,1,1} = (-1)^{j(t+1)+1} \overline{x_1 z^j} \frac{v}{x_1}$. If $v = (x_0 x_1)^i$, then $u = x_1 (x_0 x_1)^j$ for some $0 \leq j \leq k-i-1$, and we have $a_{u,w,1,1} = (-1)^{j(t+1)+1} \overline{x_1 z^j} \frac{x_1^*}{v^*}$.

3. If $\frac{u}{x_\beta} v \frac{w^*}{u} = 1^*$, then $wu = \frac{u}{x_\beta} v \neq 0$. In this case we have $wx_\beta \neq 0$. If $2 \nmid l_v = l_w + 1$, then we have $w = (x_\beta x_{\beta+1})^i$ for some $1 \leq i \leq k-1$. Then we have either $u = x_\beta$, $v = x_\beta (x_{\beta+1} x_\beta)^i$ or $u = (x_\beta x_{\beta+1})^{k-i}$, $v = x_\beta (x_{\beta+1} x_\beta)^i$. We have $\frac{w}{x_\alpha} \neq 0$ only for $\alpha = \beta$ in this case. Note also that $\frac{1^*}{x_\beta} \frac{w}{x_\beta} = 0$. Consequently, $a_{u,w,\beta,\beta}$ can be nonzero only for $\beta = 1$. Hence, we have to consider only the case where $w = (x_1 x_0)^i$, $v = x_1 (x_0 x_1)^i$. Now we have $a_{x_1, (x_1 x_0)^i, 1, 1} = a_{(x_1 x_0)^{k-i}, (x_1 x_0)^i, 1, 1} = (-1)^{(j+1)t+1} \overline{x_1 x_0 z^j} x_0 (x_1 x_0)^{i-1}$.

Let now $2 \mid l_v = l_w + 1$. Then we have $w = x_{\beta+1} (x_\beta x_{\beta+1})^{i-1}$ for some $1 \leq i \leq k$. In this case $\frac{w}{x_\alpha} \neq 0$ only for $\alpha = \beta + 1$ and $a_{u,w,\alpha,\beta}$ can be nonzero only for $\beta = 1$, $\alpha = 0$. If $v = (x_1 x_0)^i$ for some $i < k$, then $u = (x_1 x_0)^j$ for some $1 \leq j \leq k-i$ and we have $a_{u,w,0,1} = (-1)^{(j+1)t+1} \overline{x_1 x_0 z^j} (x_1 x_0)^{i-1}$. If $v = (x_0 x_1)^i$ for some $i \leq k$, then $u = x_1 (x_0 x_1)^j$ for some $0 \leq j \leq k-i$ and we have $a_{u,w,0,1} = (-1)^{(j+1)t+1} \overline{x_1 x_0 z^j} (x_1 x_0)^{i-1}$.

Putting all the obtained equalities together we obtain the statement of the lemma. \square

Lemma 21. *If $v \in G$, $\alpha \in \{0, 1\}$, and j and t are some integers, then*

$$\begin{aligned}
& Tr(\mu_R St_L) \sigma_{P,P}(v \otimes \overline{z^t} \otimes \overline{z^{j-t}}) \\
&= \begin{cases} 0, & \text{if } v = x_0, t > 0; \\ (-1)^{jt+j+1} \otimes \overline{x_\alpha z^j}, & \text{if } v = x_\alpha \text{ and either } t = 0 \text{ or } \alpha = 1; \\ (-1)^{jt+j+1} (x_0 x_1)^i \otimes \overline{x_0 z^j}, & \text{if } v = x_0 (x_1 x_0)^i, 1 \leq i \leq k-1 \text{ and } t > 0; \\ (-1)^{jt+j+1} ((x_\alpha x_{\alpha+1})^i + (x_{\alpha+1} x_\alpha)^i) \otimes \overline{x_\alpha z^j}, & \text{if } v = x_\alpha (x_{\alpha+1} x_\alpha)^i, 1 \leq i \leq k-1 \\ & \text{and either } t = 0 \text{ or } \alpha = 1; \\ (-1)^{(j+1)(t+1)} (i-1) x_1 (x_0 x_1)^{i-1} \otimes \overline{x_0 z^j} \\ + (-1)^{(j+1)t+1} i x_0 (x_1 x_0)^{i-1} \otimes \overline{x_1 z^j}, & \text{if } v = (x_0 x_1)^i, 1 \leq i \leq k, t > 0; \\ i((-1)^{j+1} x_1 (x_0 x_1)^{i-1} \otimes \overline{x_0 z^j} \\ - x_0 (x_1 x_0)^{i-1} \otimes \overline{x_1 z^j}), & \text{if } v = (x_0 x_1)^i, 1 \leq i \leq k, t = 0; \\ i((-1)^{(j+1)(t+1)} x_0 (x_1 x_0)^{i-1} \otimes \overline{x_1 z^j} \\ + (-1)^{(j+1)t+1} x_1 (x_0 x_1)^{i-1} \otimes \overline{x_0 z^j}), & \text{if } v = (x_1 x_0)^i, 0 \leq i \leq k-1. \end{cases}
\end{aligned}$$

Proof. The case $t = 0$ is clear. Assume now that $t > 0$. Let us introduce the notation $a_{u,w,\alpha,\beta} := \mu_R St_L \left(\frac{u^*}{v^*} \overline{x_\alpha z^{j-t}} \frac{u}{x_\alpha} w^* \otimes \overline{x_\beta z^{t-1}} \frac{w}{x_\beta} \right)$. Then

$$\begin{aligned}
& Tr(\mu_R St_L) \sigma_{P,P}(v \otimes \overline{z^t} \otimes \overline{z^{j-t}}) \\
&= \begin{cases} 1 \otimes \sum_{\substack{u, w \in G \\ \alpha, \beta \in \{0, 1\}}} (-1)^{j(l_u+l_v+1)+t(l_u+l_w+l_v+1)+\beta+1} a_{u,w,\alpha,\beta}, & \text{if } v \neq 1^*; \\ 1 \otimes \sum_{\substack{u, w \in G \\ \beta \in \{0, 1\}}} (-1)^{j(l_u+1)+t(l_u+l_w+1)+\beta+1} a_{u,w,l_u,\beta}, & \text{if } v = 1^*; \end{cases}
\end{aligned}$$

It follows from Lemmas 15 and 16 that if $a_{u,w,\alpha,\beta} \neq 0$, then $\frac{u}{x_\alpha} w^* \in \{x_\alpha, x_1^*, 1^*\}$. Let us consider each of the mentioned values separately.

1. If $\frac{u}{x_\alpha} w^* = x_\alpha$, then one can show that $u = w^* = x_\alpha$. Since in this case $\frac{w}{x_\alpha} = 0$ and $a_{u,w,\alpha,\beta}$ can be nonzero only for $\beta = \alpha$, we obtain $a_{u,w,\alpha,\beta} = 0$.

2. If $\frac{u}{x_\alpha} w^* = x_1^*$, then $a_{u,w,\alpha,\beta}$ can be nonzero only for $\alpha = 1$. In this case we have $u = w$, $\frac{w}{x_1} \neq 0$. If $\beta \neq 1$, then $\frac{w}{x_\beta} = 0$. Now we have $1 \otimes a_{u,u,1,1} = (-1)^{jt+1} \frac{u}{x_1} \frac{u^*}{v^*} \otimes \overline{x_1 z^j} \in Tr(P)$. If $v = x_0 (x_1 x_0)^i$ for $0 \leq i \leq k-1$, then $\frac{u}{x_1} \frac{u^*}{v^*} = 0$ for all $u \in G$. If $v = x_1 (x_0 x_1)^i$ for $0 \leq i \leq k-1$, then $\frac{u}{x_1} \frac{u^*}{v^*} = (x_1 x_0)^i$ for $u = x_1$, $\frac{u}{x_1} \frac{u^*}{v^*} = (x_0 x_1)^i$ for $u = v$, and $\frac{u}{x_1} \frac{u^*}{v^*} = 0$ for all other $u \in G$. If $v = (x_0 x_1)^i$ for $1 \leq i \leq k$, then $\frac{u}{x_1} \frac{u^*}{v^*} = x_0 (x_1 x_0)^{i-1}$ for $u = x_1 (x_0 x_1)^j$, $0 \leq j \leq i-1$, and $\frac{u}{x_1} \frac{u^*}{v^*} = 0$ for all other $u \in G$, except the case $i = k$, $u = (x_1 x_0)^j$, $1 \leq j \leq k$ that does not occur. If $v = (x_1 x_0)^i$ for $0 \leq i \leq k-1$, then $\frac{u}{x_1} \frac{u^*}{v^*} = x_0 (x_1 x_0)^{i-1}$ for $u = (x_1 x_0)^j$, $1 \leq j \leq i$, and $\frac{u}{x_1} \frac{u^*}{v^*} = 0$ for all other $u \in G$.

3. If $\frac{u}{x_\alpha} w^* = 1^*$, then $w = \frac{u}{x_\alpha}$. If $\beta = \alpha$, then $\frac{w}{x_\beta} = 0$. Hence, $a_{u,\frac{u}{x_\alpha},\alpha,\beta}$ can be nonzero only for $\alpha = 1$, $\beta = 0$. Now we have $1 \otimes a_{u,\frac{u}{x_1},1,0} = (-1)^{jt+j+t} \frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 \otimes \overline{x_1 z^j} \in Tr(P)$. If $v = x_0 (x_1 x_0)^i$ for $0 \leq i \leq k-1$, then $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = (x_0 x_1)^i$ for $u = x_1 x_0$, $i > 0$, and $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = 0$ for all other $u \in G$. If $v = x_1 (x_0 x_1)^i$ for $0 \leq i \leq k-1$, then $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = 0$ for all $u \in G$. If $v = (x_0 x_1)^i$ for $1 \leq i \leq k$, then $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = x_1 (x_0 x_1)^{i-1}$ for $u = x_1 (x_0 x_1)^j$,

$1 \leq j \leq i-1$, and $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = 0$ for all other $u \in G$, except the case $i = k$, $u = (x_1 x_0)^j$, $1 \leq j \leq k$ that does not occur. If $v = (x_1 x_0)^i$ for $0 \leq i \leq k-1$, then $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = x_1 (x_0 x_1)^{i-1}$ for $u = (x_1 x_0)^j$, $1 \leq j \leq i$, and $\frac{u}{x_1 x_0} \frac{u^*}{v^*} x_1 = 0$ for all other $u \in G$.

Putting all the obtained equalities together we obtain the statement of the lemma. \square

Let us define the map $\mathcal{B}_P : Tr(P) \rightarrow Tr(P[1])$ by the following equalities:

$$\mathcal{B}_P(1 \otimes \bar{a}) = 0 \quad (a \in B);$$

$$\mathcal{B}_P(x_\alpha(x_{\alpha+1}x_\alpha)^i \otimes \bar{z^j}) = \begin{cases} 0, & \text{if } 0 \leq i \leq k-1, 2 \nmid j; \\ 1 \otimes \overline{x_\alpha z^j}, & \text{if } i = 0, 2 \mid j; \\ ((x_\alpha x_{\alpha+1})^i + (x_{\alpha+1} x_\alpha)^i) \otimes \overline{x_\alpha z^j}, & \text{if } 1 \leq i \leq k-1, 2 \mid j; \end{cases}$$

$$\mathcal{B}_P((x_\alpha x_{\alpha+1})^i \otimes \bar{z^j}) = (jk + i)((-1)^j x_{\alpha+1} (x_\alpha x_{\alpha+1})^{i-1} \otimes \overline{x_\alpha z^j} + x_\alpha (x_{\alpha+1} x_\alpha)^{i-1} \otimes \overline{x_{\alpha+1} z^j}) \quad (1 \leq i \leq k-1);$$

$$\mathcal{B}_P(1^* \otimes \bar{z^j}) = (j+1)k((-1)^j x_0^* \otimes \overline{x_0 z^j} + x_1^* \otimes \overline{x_1 z^j});$$

$$\mathcal{B}_P(x_{\alpha+1}(x_\alpha x_{\alpha+1})^i \otimes \overline{x_\alpha^p z^j}) = 0 \quad (0 \leq i \leq k-2);$$

$$\mathcal{B}_P(x_\alpha^* \otimes \overline{x_\alpha^p z^j}) = \begin{cases} (-1)^{\alpha+1} \otimes \overline{x_\alpha^{p-1} z^{j+1}}, & \text{if } 2 \mid p, 2 \mid j; \\ 0, & \text{if } 2 \mid p, 2 \nmid j; \\ (-1)^{(j+1)(\alpha+1)}(j+1) \otimes \overline{x_\alpha^{p-1} z^{j+1}}, & \text{if } 2 \nmid p; \end{cases}$$

$$\mathcal{B}_P(x_\alpha \otimes \overline{x_\alpha^p z^j}) = \begin{cases} (p+1) \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \mid p, 2 \mid j; \\ 0, & \text{if } 2 \mid p, 2 \nmid j; \\ (-1)^{\alpha+1} j \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \nmid p, 2 \mid j; \\ (j+p+1) \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \nmid p, 2 \nmid j; \end{cases}$$

$$\mathcal{B}_P(x_\alpha(x_{\alpha+1}x_\alpha)^i \otimes \overline{x_\alpha^p z^j}) = \begin{cases} ((x_{\alpha+1}x_\alpha)^i + (x_\alpha x_{\alpha+1})^i) \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \mid p, 2 \mid j; \\ 0, & \text{if } 2 \mid p, 2 \nmid j; \\ (j+1)((-1)^{j+1}(x_{\alpha+1}x_\alpha)^i + (x_\alpha x_{\alpha+1})^i) \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \nmid p; \end{cases} \quad (1 \leq i \leq k-1);$$

$$\mathcal{B}_P((x_{\alpha+1}x_\alpha)^i \otimes \overline{x_\alpha^p z^j}) = \begin{cases} (j+1)x_{\alpha+1}(x_\alpha x_{\alpha+1})^{i-1} \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \mid p; \\ x_{\alpha+1}(x_\alpha x_{\alpha+1})^{i-1} \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \nmid p, 2 \mid j; \\ 0, & \text{if } 2 \nmid p, 2 \nmid j; \end{cases} \quad (1 \leq i \leq k-1);$$

$$\mathcal{B}_P((x_\alpha x_{\alpha+1})^i \otimes \overline{x_\alpha^p z^j}) = \begin{cases} (-1)^j(j+1)x_{\alpha+1}(x_\alpha x_{\alpha+1})^{i-1} \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \mid p; \\ -x_{\alpha+1}(x_\alpha x_{\alpha+1})^{i-1} \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \nmid p, 2 \mid j; \\ 0, & \text{if } 2 \nmid p, 2 \nmid j; \end{cases} \quad (1 \leq i \leq k-1);$$

$$\mathcal{B}_P(1^* \otimes \overline{x_\alpha^p z^j}) = \begin{cases} (-1)^\alpha(j+1)x_\alpha \otimes \overline{x_\alpha^{p-1} z^{j+1}} + (j+p+1)x_\alpha^* \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \mid p, 2 \mid j; \\ (j+1)(x_\alpha \otimes \overline{x_\alpha^{p-1} z^{j+1}} + (-1)^\alpha x_\alpha^* \otimes \overline{x_\alpha^{p+1} z^j}), & \text{if } 2 \mid p, 2 \nmid j; \\ -x_\alpha \otimes \overline{x_\alpha^{p-1} z^{j+1}} + p x_\alpha^* \otimes \overline{x_\alpha^{p+1} z^j}, & \text{if } 2 \nmid p, 2 \mid j; \\ 0, & \text{if } 2 \nmid p, 2 \nmid j. \end{cases}$$

Theorem 22. *The map \mathcal{B}_P induces the Connes' differential on $\mathrm{HH}_*(A)$.*

Proof. Follows from Lemmas 15–21 □

Note that $\mathrm{Hom}_{A^e}(P_n, A) \cong \mathrm{Hom}_{\mathbf{k}}(B_n, A) \cong A^{\dim_{\mathbf{k}} B_n} = A^{n+1}$. We choose this isomorphism in the following way. We send $f \in \mathrm{Hom}_{A^e}(P_n, A)$ to

$$\begin{cases} \sum_{\substack{p+2j=n, \\ p>0, \alpha \in \{0,1\}}} f(\overline{x_\alpha^p z^j}) e_{p+\alpha}^n, & \text{if } 2 \nmid n, \\ f(\overline{z^{\frac{n}{2}}}) e_1^n + \sum_{\substack{p+2j=n, \\ p>0, \alpha \in \{0,1\}}} f(\overline{x_\alpha^p z^j}) e_{p+\alpha}^n, & \text{if } 2 \mid n. \end{cases}$$

Here $e_i^n \in A^{n+1}$ is such an element that $\pi_j^n(e_i) = 0$ for $j \neq i$ and $\pi_i^n(e_i) = 1$, where $\pi_j^n : A^{n+1} \rightarrow A$ ($1 \leq j \leq n+1$) is the canonical projection on the j -th component of the direct sum. We identify $\mathrm{Hom}_{A^e}(P_n, A)$ and $A^{\dim_{\mathbf{k}} B_n}$ by the just defined isomorphism.

Let us introduce some elements of $\mathrm{Hom}_{A^e}(P, A) = \bigoplus_{n \geq 0} A^{\dim_{\mathbf{k}} B_n}$.

- $p_1 = x_0 x_1 + x_1 x_0$, $p_2 = x_1^*$, $p_2' = x_0^*$ and $p_3 = 1^*$ are elements of $\mathrm{Hom}_{A^e}(P_0, A) = A$;
- $u_1 = (x_0, 0)$, $u_1' = (0, x_1)$, $u_2 = (1, 0)$, $u_2' = (0, 1)$ are elements of $\mathrm{Hom}_{A^e}(P_1, A) = A^2$;
- $v = (1, 0, 0)$, $v_1 = (x_0 x_1 - x_1 x_0, 0, 0)$, $v_2 = (0, 1, 0)$, $v_2' = (0, 0, 1)$, $v_3 = (1^*, 0, 0)$ are elements of $\mathrm{Hom}_{A^e}(P_2, A) = A^3$;

- $w_1 = (x_0, 0, 0, 0)$, $w_2 = (x_0^*, 0, 0, 0)$, $w'_2 = (0, x_1^*, 0, 0)$ are elements of $\text{Hom}_{A^e}(P_3, A) = A^4$;
- $t = (1, 0, 0, 0, 0)$ is an elements of $\text{Hom}_{A^e}(P_4, A) = A^5$.

It is proved in [6, 7] that the algebra $\text{HH}^*(A)$ is generated by the cohomological classes of the elements from \mathcal{X} , where

$$\mathcal{X} = \begin{cases} \{p_1, p_2, p'_2, p_3, u_1, u'_1, u_2, u'_2, v\}, & \text{if char } \mathbf{k} = 2; \\ \{p_1, p_2, p'_2, u_1, u'_1, v_1, v_2, v'_2, v_3, w_1, w_2, w'_2, t\}, & \text{if char } \mathbf{k} \neq 2, \text{ char } \mathbf{k} \mid k; \\ \{p_1, p_2, p'_2, u_1, u'_1, v_1, v_2, v'_2, t\}, & \text{if char } \mathbf{k} \neq 2, \text{ char } \mathbf{k} \nmid k. \end{cases}$$

Note that our notation is essentially the same as the notation of [6], but slightly differs from the notation of [7]. For the simplicity we denote the cohomological class of $a \in \text{Hom}_{A^e}(P_n, A)$ by a too.

It follows from the previous section that we can define the BV differential $\mathcal{D}_P : \text{HH}^*(A) \rightarrow \text{HH}^*(A)$ by the formula

$$\mathcal{D}_P(f)(a) = \sum_{v \in G} \theta \text{Tr}(f) \mathcal{B}_P(v \otimes a) v^*$$

for $a \in P$. One can show that

$$\begin{aligned} \mathcal{D}_P(u_2) &= \mathcal{D}_P(p'_2 u_2) = \mathcal{D}_P(u_2^2) = \mathcal{D}_P(u'_2) = \mathcal{D}_P(p_2 u'_2) = \mathcal{D}_P((u'_2)^2) = \mathcal{D}_P(v) = \mathcal{D}_P(p_1 v) \\ &= \mathcal{D}_P(u_1 v) = \mathcal{D}_P(u'_1 v) = \mathcal{D}_P(u_2 v) = \mathcal{D}_P(u'_2 v) = \mathcal{D}_P(v^2) = \mathcal{D}_P(v_1) = \mathcal{D}_P(p_1 v_1) = \mathcal{D}_P(v_2) \\ &= \mathcal{D}_P(p'_2 v_2) = \mathcal{D}_P(v_2^2) = \mathcal{D}_P(v'_2) = \mathcal{D}_P(p_2 v'_2) = \mathcal{D}_P((v'_2)^2) = \mathcal{D}_P(w_1) = \mathcal{D}_P(t) = \mathcal{D}_P(p_1 t) \\ &= \mathcal{D}_P(v_1 t) = \mathcal{D}_P(v_2 t) = \mathcal{D}_P(v'_2 t) = \mathcal{D}_P(w_1 t) = \mathcal{D}_P(t^2) = 0, \end{aligned}$$

$$\begin{aligned} \mathcal{D}_P(u_1) &= \mathcal{D}_P(u'_1) = k, \mathcal{D}_P(p_1 u_1) = (k-1)p_1, \mathcal{D}_P(p_2 u'_1) = p_2, \mathcal{D}_P(p'_2 u_1) = p'_2, \\ \mathcal{D}_P(v_3) &= u'_1 - u_1, \mathcal{D}_P(p_2 v) = u'_2, \mathcal{D}_P(p'_2 v) = u_2, \mathcal{D}_P(p_3 v) = u_1 + u'_1, \\ \mathcal{D}_P(u_1 u'_1) &= k(u'_1 - u_1), \mathcal{D}_P(u_1 u_2) = k u_2, \mathcal{D}_P(u'_1 u'_2) = k u'_2, \mathcal{D}_P(w_2) = v_2, \\ \mathcal{D}_P(w'_2) &= -v'_2, \mathcal{D}_P(u_1 v_1) = (2k-1)v_1, \mathcal{D}_P(u_1 v_2) = (k+2)v_2, \mathcal{D}_P(u_1 v'_2) = k v'_2, \\ \mathcal{D}_P(u'_1 v_2) &= k v_2, \mathcal{D}_P(u'_1 v'_2) = (k+2)v'_2, \mathcal{D}_P(v_2 v_3) = 3w_2, \mathcal{D}_P(v'_2 v_3) = 3w'_2, \\ \mathcal{D}_P(u_1 t) &= \mathcal{D}_P(u'_1 t) = 3kt, \mathcal{D}_P(v_2 w_2) = v_2^2, \mathcal{D}_P(v'_2 w'_2) = -(v'_2)^2, \\ \mathcal{D}_P(v_3 t) &= 3(u'_1 - u_1)t, \mathcal{D}_P(w_2 t) = 3v_2 t, \mathcal{D}_P(w'_2 t) = 3v'_2 t. \end{aligned}$$

We use here the results of [6, 7]. In particular, we use the formulas for some products in $\text{HH}^*(A)$ and the description of some coboundaries. Alternatively, one can use the formula $f \smile g = (f \otimes g) S \eta_L$ and Lemmas 15 and 16 to compute products in $\text{HH}^*(A)$. Note also that in each of the formulas above we assume that all the elements included in the formula lie in \mathcal{X} . For example, if v appears in some equality, then this equality holds for $\text{char } \mathbf{k} = 2$, but doesn't have to hold for $\text{char } \mathbf{k} \neq 2$. We also have $\mathcal{D}_P(a) = 0$ for all $a \in \text{HH}^0(A)$. Now it is not hard to recover the Gerstenhaber bracket and the rest of the BV differential on the Hochschild cohomology of A using relations between the generators of $\text{HH}^*(A)$ described in [6, 7] and the graded Leibniz rule for the Gerstenhaber bracket and the cup product.

Remark 3. We think that the Gerstenhaber bracket and even the BV differential on $\text{HH}^*(A)$ can be computed in some simpler way using different tricks. But the aim of this example is to show that our formulas are reasonable for a direct application.

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